ON THE HELMHOLTZ DECOMPOSITION IN WEAKLY ANISOTROPIC VTI MEDIA

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ABSTRACT

We derive an extension of the Helmholtz decomposition theorem for vector fields to represent the displacement wavefield in a homogeneous weakly anisotropic VTI medium. Besides an elegant generalization of known facts about wave propagation in VTI media, the decomposition enables us to derive a system of partial differential equations to describe the propagation of pseudo-acoustic waves. We also obtain the parabolic approximation for the pseudo-acoustic wave equation providing a rigorous demonstration of dispersion relations established heuristically by Alkhalifah (2000). Finally, we show how the decomposition can be applied to perform wave-mode separation in VTI media.

INTRODUCTION

The anisotropic elastic wave equation is an important tool to describe seismic wave propagation. However, this partial differential equation is difficult to solve even in homogeneous media. In the isotropic case, the Helmholtz decomposition allows to decouple compressional (P wave) and shear (S wave) components and find separate equations to describe each wave mode independently. For the general anisotropic case, such a decoupling is not possible, i.e., the anisotropic equation must be solved at once.

Since the Helmholtz decomposition has important applications in practice, approximate wavefield decoupling methods are still a subject of ongoing investigations. Several authors have proposed ideas to separate wavefields of so-called quasi-P and quasi-S waves. Dellinger and Etgen (1990) suggested separating the wavefield in an anisotropic medium by projection of the wavefield in directions where the q-P and q-S waves are polarised. For heterogeneous media Yan and Sava (2009) suggested using pseudo-operators to separated the wavefield. His technique is based on solving the Christoffel equation.

In last year's WIT report, Bloot et al. (2011) derived an elastic wave equation for a weakly anisotropic medium with vertical transverse isotropy (VTI), directly based on the parameterisation of such a medium by Thomsen (1986). Starting from this equation, we propose an extension of Helmholtz decomposition to represent the elastic wavefield in weakly anisotropic VTI media. The decomposition simplifies the derivation of solutions for the wave equation in such media. Additionally, it permits to construct operators for wave-mode separation.

As a result, we show that the elastic wave equation can be transformed into two simple differential equations and a system of two coupled differential equations. The coupled equations system describes the pseudo-acoustic q-P wavefield and the other two equations represent the q-SV and q-SH wavefields, respectively. We study these waves looking at issues such as coupling. In this way, we show how a proper choice of the source term representation can be used to decouple wave modes.

In addition to the extended Helmholtz decomposition, we also derive a parabolic approximation for the pseudo-acoustic wave equation that is free of artifacts. This equation can be used for wave-equation migration. Moreover, we provide a rigorous derivation of the dispersion relation that Alkhalifah (2000) used to formulate his pseudo-acoustic wave equation.

VTI ELASTIC WAVE EQUATION

Our starting point is the elastic wave equation for a heterogeneous VTI media with weak anisotropy (Bloot et al., 2011),

$$\begin{aligned}\rho\frac{\partial^2}{\partial t^2}\mathbf{u}=\mathbf{f} \quad &+ \quad \nabla\cdot[\mu\nabla\mathbf{u}]+\nabla[(\lambda+\mu)\nabla\cdot\mathbf{u}]+\nabla\mu\times(\nabla\times\mathbf{u})+2(\nabla\mu\cdot\nabla)\mathbf{u}\\ &+ \quad \widehat{\nabla}[\delta(\lambda+2\mu)\nabla\cdot\mathbf{u}]+\nabla[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}]-2\widehat{\nabla}[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}]\\ &+ \quad 2\widehat{\nabla}^{\perp}[\mu\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}]+4\mathbf{J}\nabla\mathbf{u}\widehat{\nabla}^{\perp}[\mu\gamma]+\widehat{\nabla}[\epsilon(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}], \end{aligned} \tag{1}$$

where $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is the displacement vector, $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$ is the source term, ρ is density, λ and μ are the Lamé parameters, and ϵ, δ and γ are the anisotropy parameters of Thomsen (Thomsen, 1986), with $|\delta|\ll 1$. No restrictions apply to ϵ and γ. We have also used the notations,

$$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)^T,\quad \widehat{\nabla}\equiv\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},0\right)^T,\quad \widehat{\nabla}^{\perp}\equiv\mathbf{J}\widehat{\nabla}=\left(\frac{\partial}{\partial y},-\frac{\partial}{\partial x},0\right)^T, \tag{2}$$

with

$$\mathbf{J}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}, \tag{3}$$

and the elements of the Jacobian matrix $\nabla\mathbf{u}$ are the partial derivatives of u_i $(i=1,2,3)$ with respect to the Cartesian coordinates. Equation (1) describes a system of three coupled differential equations, because all components of $\mathbf{u}$ appear in all three equations of system (1).

In the case of a homogeneous VTI medium, equation (1) reduces to

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathbf{u} \quad &= \quad \frac{1}{\rho}\mathbf{f}+\alpha^2\nabla(\nabla\cdot\mathbf{u})-\beta^2\nabla\times\nabla\times\mathbf{u}+2\alpha^2(\epsilon-\delta)\widehat{\nabla}(\widehat{\nabla}\cdot\mathbf{u})\\ &\quad +\delta\alpha^2\widehat{\nabla}(\nabla\cdot\mathbf{u})+\delta\alpha^2\nabla(\widehat{\nabla}\cdot\mathbf{u})+2\beta^2\gamma\widehat{\nabla}^{\perp}(\widehat{\nabla}^{\perp}\cdot\mathbf{u}),\end{aligned} \tag{4}$$

where α and β represents the velocities of the q-P and q-SV waves along the axis of symmetry and are given by (Tsvankin, 2001)

$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}\quad\text{and}\quad\beta=\sqrt{\frac{\mu}{\rho}}. \tag{5}$$

If $\delta=\gamma=\epsilon=0$, equation (4) reduces to the well-known isotropic form

$$\frac{\partial^2}{\partial t^2}\mathbf{u}=\frac{1}{\rho}\mathbf{f}+\alpha^2\nabla(\nabla\cdot\mathbf{u})-\beta^2\nabla\times\nabla\times\mathbf{u}, \tag{6}$$

which can be separated by Helmholtz decomposition using scalar and vector potentials.

The Helmholtz decomposition ensures that for any vector field $\mathbf{\Omega}$ there exist a scalar potential ψ and a vector potential $\mathbf{\Psi}$ so that

$$\mathbf{\Omega}=\nabla\psi+\nabla\times\mathbf{\Psi}\quad\text{with}\quad\nabla\cdot\mathbf{\Psi}=0. \tag{7}$$

Upon the use of decomposition (7), it is possible to solve equation (6) analytically. This procedure demonstrates that P and S waves decouple in homogeneous, isotropic, elastic media, allowing to study them independently. For more details see Pujol (2003) and Aki and Richards (1980).

In this paper we show that the homogeneous VTI wave equation (4) can be solved in a similar way. For this purpose, we show that the wavefield in a VTI medium can be decomposed into other types of potential fields by using an extension of the Helmholtz theorem. The idea is to find a decomposition for a general vector field and verify that the vector field for a VTI medium can be written in an identical manner. The generalized Helmholtz decomposition relies on the fact that for an arbitrary vector field $\widehat{\mathbf{\Omega}}=(\Omega_1,\Omega_2,0)^T$, there exist scalar functions ϕ and Φ such that arbitrary vector field $\widehat{\mathbf{\Omega}}$

$$\widehat{\mathbf{\Omega}}=\widehat{\nabla}\phi+\widehat{\nabla}^{\perp}\Phi. \tag{8}$$

It is immediately visible that that identity (8) is valid if there is a vector field $\mathbf{W} = (W_1, W_2, 0)^T$, such that $\widehat{\mathbf{\Omega}}$ can be written as

$$\widehat{\mathbf{\Omega}} = \partial_{xx}\mathbf{W} + \partial_{yy}\mathbf{W} \ . \tag{9}$$

because

$$\widehat{\mathbf{\Omega}} = \partial_{xx}\mathbf{W} + \partial_{yy}\mathbf{W} = \widehat{\nabla}\left(\widehat{\nabla}\cdot\mathbf{W}\right) + \widehat{\nabla}^{\perp}\left(\widehat{\nabla}^{\perp}\cdot\mathbf{W}\right) \ . \tag{10}$$

In this case, we simply need to choose $\phi = \widehat{\nabla}\cdot\mathbf{W}$ and $\Phi = \widehat{\nabla}^{\perp}\cdot\mathbf{W}$ to achieve decomposition (8). Therefore, since it is always possible to find a solution $\mathbf{W}$ to the two-dimensional Poisson equation (9), the decomposition is valid for any arbitrary vector field $\widehat{\mathbf{\Omega}}$. Decomposition (8) can be viewed as an extension of Helmholtz theorem for horizontal vector fields.

Together, decompositions (7) and (8) provide us with the means to obtain a generalized Helmholtz decomposition for vector wavefields in VTI media. For this purpose, let us consider an arbitrary vector field $\mathbf{u}$ and write it as a sum of two fields $\mathbf{\Omega}$ and $\widehat{\mathbf{\Omega}}$, i.e., $\mathbf{u} = \mathbf{\Omega} + \widehat{\mathbf{\Omega}}$. Of course, several such sums exist, for example, $\mathbf{\Omega} = \mathbf{u}$ and $\widehat{\mathbf{\Omega}} = \mathbf{0}$, or $\mathbf{\Omega} = (0, 0, u_3)^T$ and $\widehat{\mathbf{\Omega}} = (u_1, u_2, 0)^T$. We know from equations (7) and (8) that there exist scalar potentials ψ, ϕ and Φ, and a vector potential $\mathbf{\Psi}$ such that

$$\mathbf{u} = \nabla\psi + \nabla\times\mathbf{\Psi} + \widehat{\nabla}\phi + \widehat{\nabla}^{\perp}\Phi \quad \text{with} \quad \nabla\cdot\mathbf{\Psi} = 0. \tag{11}$$

This is the generalization of Helmholtz decomposition which is clearly not unique for a fixed field $\mathbf{u}$, but it is unique for fixed $\mathbf{\Omega}$ and $\widehat{\mathbf{\Omega}}$, such that $\mathbf{u} = \mathbf{\Omega} + \widehat{\mathbf{\Omega}}$. To impose uniqueness, we can still require one additional property for the set of potential fields ϕ, Φ, ψ, and $\mathbf{\Psi}$. We choose the condition

$$\widehat{\nabla}\cdot(\nabla\times\mathbf{\Psi}) = 0. \tag{12}$$

To solve equation (4) using decomposition (11), we first consider the source term to be decomposed accordingly as

$$\mathbf{f} = \nabla\psi_f + \nabla\times\mathbf{\Psi}_f + \widehat{\nabla}\phi_f + \widehat{\nabla}^{\perp}\Phi_f \quad \text{with} \quad \nabla\cdot\mathbf{\Psi}_f = 0. \tag{13}$$

with three scalar potentials ϕ_f, Φ_f, and ψ_f, and one vector potential $\mathbf{\Psi}_f$. Again, this decomposition is unique under the condition

$$\widehat{\nabla}\cdot(\nabla\times\mathbf{\Psi}_f) = 0. \tag{14}$$

Substituting expression (13) in (4), we find

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathbf{u} \quad = \quad & \frac{1}{\rho}\nabla\psi_f + \frac{1}{\rho}\nabla\times\mathbf{\Psi}_f + \frac{1}{\rho}\widehat{\nabla}\phi_f + \frac{1}{\rho}\widehat{\nabla}^{\perp}\Phi_f + \alpha^2\nabla(\nabla\cdot\mathbf{u} + \delta\widehat{\nabla}\cdot\mathbf{u}) \\ & - \beta^2\nabla\times\nabla\times\mathbf{u} + \alpha^2\widehat{\nabla}(\delta\nabla\cdot\mathbf{u} + 2(\epsilon-\delta)\widehat{\nabla}\cdot\mathbf{u}) + \beta^2\widehat{\nabla}^{\perp}(2\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}). \end{aligned} \tag{15}$$

Double integration of this equation in the variable t immediately leads to an expression for $\mathbf{u}$ of the form of equation (11), with

$$\psi = \alpha^2\int_0^t\int_0^\tau\left[\left(\nabla\cdot\mathbf{u} + \delta\widehat{\nabla}\cdot\mathbf{u}\right) + \frac{1}{\rho\alpha^2}\psi_f\right]d\tau' d\tau, \tag{16}$$

$$\mathbf{\Psi} = -\beta^2\int_0^t\int_0^\tau\left[\nabla\times\mathbf{u} - \frac{1}{\rho\beta^2}\mathbf{\Psi}_f\right]d\tau' d\tau, \tag{17}$$

$$\phi = \alpha^2\int_0^t\int_0^\tau\left[\left(\delta\nabla\cdot\mathbf{u} + 2(\epsilon-\delta)\widehat{\nabla}\cdot\mathbf{u}\right) + \frac{1}{\rho\alpha^2}\phi_f\right]d\tau' d\tau, \tag{18}$$

and

$$\Phi = \beta^2\int_0^t\int_0^\tau\left[\left(2\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}\right) + \frac{1}{\rho\beta^2}\Phi_f\right]d\tau' d\tau. \tag{19}$$

Potential Functions

We are primarily interested in finding a solution to equation (4). Equations (11) and (16)-(19) suggest that there should be functions ψ, ϕ, Φ and $\mathbf{\Psi}$ which together form the field $\mathbf{u}$. For homogeneous VTI media theoretical (Tsvankin, 2001) and experimental work de Figueiredo et al. (2012) has shown that q-P and q-SV waves do not vary with azimuth. This observations motivated us to impose the mathematical condition

$$\widehat{\nabla} \cdot (\nabla \times \mathbf{\Psi}) = 0, \tag{20}$$

which is equivalent to

$$\frac{\partial}{\partial z}\left(\frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y}\right) = 0, \tag{21}$$

guaranteeing invariance of rotational symmetry along the vertical axis.

In this way, the generalized decomposition (11) is a true extension of the isotropic one (7) to VTI media. In the limit of weak anisotropy, it does not contradict the Lamé solution for the isotropic case. In other words, for $\epsilon, \delta, \gamma \to 0$, expressions (16)-(19) become

$$\mathbf{u} = \nabla\psi + \nabla \times \mathbf{\Psi} \quad \text{with} \quad \nabla \cdot \mathbf{\Psi} = 0, \tag{22}$$

when the anisotropic source terms vanish, i.e., $\phi_f = \Phi_f = 0$. This ensures that this technique reduces to the isotropic case when $\epsilon = \delta = \gamma = 0$.

The generalized Helmholtz decomposition (11) for VTI media leads to simpler differential equations for the wave potentials. Introducing the auxiliary notations for the radial Laplacian,

$$\Delta = \Delta_r + \frac{\partial^2}{\partial z^2}, \quad \text{with} \quad \Delta_r = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \tag{23}$$

we can formulate the

Wavefield Separation Theorem: *Consider $\mathbf{u}(\mathbf{x}, t)$ to denote a continuous vector field which is differentiable up to fourth order in $I\!R^3$ and satisfies equation (21) as well as the initial conditions*

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \quad \textit{and} \quad \mathbf{u}_t(\mathbf{x}, 0) = \mathbf{0}. \tag{24}$$

Then

$$\mathbf{u}(\mathbf{x}, t) = \nabla\psi(\mathbf{x}, t) + \nabla \times \mathbf{\Psi}(\mathbf{x}, t) + \widehat{\nabla}\phi(\mathbf{x}, t) + \widehat{\nabla}^{\perp}\Phi(\mathbf{x}, t) \tag{25}$$

is a solution for equation (4) provided that ψ, ϕ, Φ and $\mathbf{\Psi}$ solve the following equations

$$\begin{aligned}\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\psi &= (1+\delta)\Delta_r(\psi+\phi) + \frac{\partial^2}{\partial z^2}\psi + F_1,\\ \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\phi &= (2\epsilon-\delta)\Delta_r(\psi+\phi) + \delta\frac{\partial^2}{\partial z^2}\psi + F_2;\end{aligned} \tag{26}$$

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}\Phi = (1+2\gamma)\Delta_r\Phi + \frac{\partial^2}{\partial z^2}\Phi + F_3; \tag{27}$$

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}\mathbf{\Psi} = \Delta\mathbf{\Psi} + \nabla \times \left(\widehat{\nabla}\phi + \widehat{\nabla}^{\perp}\Phi\right) + \mathbf{F_4}; \tag{28}$$

with

$$\nabla \cdot \mathbf{\Psi} = 0; \quad \textit{and} \quad \widehat{\nabla} \cdot (\nabla \times \mathbf{\Psi}) = 0. \tag{29}$$

Functions F_1, F_2, F_3 and $\mathbf{F}_4$ depend only on the source term $\mathbf{f}$.

The proof of this theorem, as well as the expressions for source functions, can be found in Appendix A.

Note that potentials ψ and ϕ are coupled. To determine them it is necessary to solve equation system (26). Also note that equation (27), which determines potential Φ that describes the q-SH wave, is completely decoupled. Once ϕ and Φ are known, equation (28) can be solved for the vector potential $\mathbf{\Psi}$.

To obtain the solutions of both equations (27) and (28) it is sufficient to study the problem of finding a scalar function χ, solution of

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi = \nu^2\Delta_r\chi + \frac{\partial^2}{\partial z^2}\chi + F, \tag{30}$$

where F is the source term, $\nu = \sqrt{1+2\gamma}$ for (27) and $\nu = 1$ for (28), and c is the wave velocity. Note that equation (30) is a wave equation for an elliptic medium. We denote by G the Green's function of the above problem, i.e., the solution for the case $F(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t)$. Below, we will show that

$$G(\mathbf{x}, t) = \frac{1}{4\pi\|\mathbf{d}\|\nu^2}\delta\left(t - \frac{\|\mathbf{d}\|}{c}\right) \quad \text{with} \quad \mathbf{d} = \left(\frac{x}{\nu}, \frac{y}{\nu}, z\right)^T, \tag{31}$$

and then,

$$\chi(\mathbf{x}, t) = G(\mathbf{x}, t) * F(\mathbf{x}, t), \tag{32}$$

where $*$ denotes convolution in space and time.

The above extension of Helmholtz decomposition to VTI media has important applications in seismic processing, modelling, and imaging. In the next section, we derive the two-way and one-way pseudo-acoustic wave equations for weak VTI medium. In the section afterwards, we present a new algorithm for wave-mode separation as another important application of the decomposition.

PSEUDO-ACOUSTIC APPROXIMATIONS IN VTI MEDIA

The existence of a source term determines the behaviour of the wavefield. In the isotropic case, if the wavefield is generated by an omnidirectional point source, then no S-waves are generated and there is only P-wave propagation (Aki and Richards, 1980). Unfortunately, in a VTI medium, the situation is more complicated. To analyse it, let us start by considering an omnidirectional point source in equation (4), i.e., a source term of the form $\mathbf{f}(\mathbf{x}, t) = F(t)\nabla\delta(\mathbf{x})$. Then, from equation (13), we have $\mathbf{\Psi}_f = \mathbf{0}$, $\phi_f = \Phi_f = 0$. Therefore, we see from equation (32) that Φ vanishes, i.e., the q-SH wave does not propagate. Thus, the situation is described by

$$\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\psi = (1+\delta)\Delta_r(\psi+\phi) + \frac{\partial^2}{\partial z^2}\psi + \frac{1}{\rho\alpha^2}\delta(\mathbf{x})F(t), \tag{33}$$

$$\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\phi = (2\epsilon-\delta)\Delta_r(\psi+\phi) + \delta\frac{\partial^2}{\partial z^2}\psi \tag{34}$$

and

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}\mathbf{\Psi} = \Delta\mathbf{\Psi} + \nabla\times\left(\widehat{\nabla}\phi\right). \tag{35}$$

Equation (35) indicates that if $\beta \neq 0$, the presence of a omnidirectional point source induces the propagation of a q-S wave, here described by the vector potential $\mathbf{\Psi}$.

Two-Way Pseudo-Acoustic Equation

We have just seen that in the presence of an omnidirectional point source, there is a q-S wave propagating and this fact is directly related to the presence of potential ϕ [see equation (35)]. However, for this wave to propagate, the shear-wave velocity β needs to be nonzero. To eliminate this wave, we can perform the same procedure as Alkhalifah (2000), which consists of simulating the propagation in a medium where $\beta = 0$. Then, equations (33) and (34) can be rewritten as

$$\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\varphi = (1+2\epsilon)\Delta_r\varphi + (1+\delta)\frac{\partial^2}{\partial z^2}\psi + \frac{1}{\rho\alpha^2}\delta(\mathbf{x})F(t) \tag{36}$$

and

$$\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\psi = (1+\delta)\Delta_r\varphi + \frac{\partial^2}{\partial z^2}\psi + \frac{1}{\rho\alpha^2}\delta(\mathbf{x})F(t), \tag{37}$$

where $\varphi = \psi + \phi$. Note that the above equations have the same form of the equation obtained by Duveneck et al. (2008), but in our derivation, we have relied on a linearization in δ. A fact that should be emphasised is that although our equations are represented in terms of the scalar potentials ψ and ϕ, systems (33)–(34) and (36)–(37) are equivalent. In the remainder of this work, we choose to use representation (36)–(37) because it offers a simpler way to deduce the one-way wave equation.

One-Way Approximation. We now disregard the source term in the pseudo-acoustic equation system (36)–(37). It is possible to show that the 2D wave equation in one direction is given by (see Appendix B)

$$\frac{\partial^2}{\partial z^2}\widehat{\psi} + k_z^2\widehat{\psi} = 0, \tag{38}$$

where

$$k_z = \frac{\omega}{\alpha}\sqrt{1 - \frac{m^2}{1-2\eta m^2}}, \quad m^2 = \frac{V_{NMO}^2 k_x^2}{\omega^2}, \quad V_{NMO} = \alpha\sqrt{1+2\delta}, \quad \text{and} \quad \eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{39}$$

The expression for k_z is denominated dispersion relation and coincides with the one suggested heuristically by Alkhalifah (2000). Our method presents a generalization of the isotropic case to VTI media. If $\epsilon = \delta = 0$ the dispersion relation k_z reduces to isotropic case and ψ is the classic isotropic P wave. We can use equation (38) to show that the upgoing wave is given by the solution of

$$\frac{\partial\widehat{\psi}(k_x, z, \omega)}{\partial z} = -ik_z\widehat{\psi}(k_x, z, \omega). \tag{40}$$

For an initial condition $\widehat{\psi}(k_x, z_0, \omega)$, it is easy to see that

$$\widehat{\psi}(k_x, z, \omega) = \widehat{\psi}(k_x, z_0, \omega)e^{-ik_z(z-z_0)}. \tag{41}$$

This is the unidirectional solution for a VTI medium.

Recently, Amazonas et al. (2010) used the pseudo-acoustic wave equation of Alkhalifah (2000) to develop a migration in a VTI medium by finite differences using a complex Padé approximation. Our derivation provides a sound theoretical basis to this approach.

Elliptic case. Let us now investigate the case $\epsilon = \delta$ (Helbig, 1983). Expressions (16) and (18) allow us to recognise that

$$\delta\psi = \phi + \mathcal{O}(\delta^2). \tag{42}$$

Taking into account that we are neglecting terms of second order in δ, we can consider $\phi = \delta\psi$. Therefore, $\varphi = \psi + \phi = (1+\delta)\psi$. Substituting this relationship into equation (37), we find under the linear approximation $(1+\delta)^2 \approx 1 + 2\delta$ (see Appendix B)

$$\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\psi = (1+2\delta)\Delta_r\psi + \frac{\partial^2}{\partial z^2}\psi + \frac{1}{\rho\alpha^2}\delta(\mathbf{x})F(t). \tag{43}$$

The solution to this problem can be found directly from (31)–(32), with $\nu = \sqrt{1+2\delta}$ and $c = \alpha$. Thus,

$$\psi(\mathbf{x}, t) = \frac{1}{4\pi\|d\|(1+2\delta)}\frac{1}{\rho\alpha^2}F\left(t - \frac{\|d\|}{\alpha}\right), \tag{44}$$

where

$$\|d\| = \sqrt{\frac{x^2}{1+2\delta} + \frac{y^2}{1+2\delta} + z^2}. \tag{45}$$

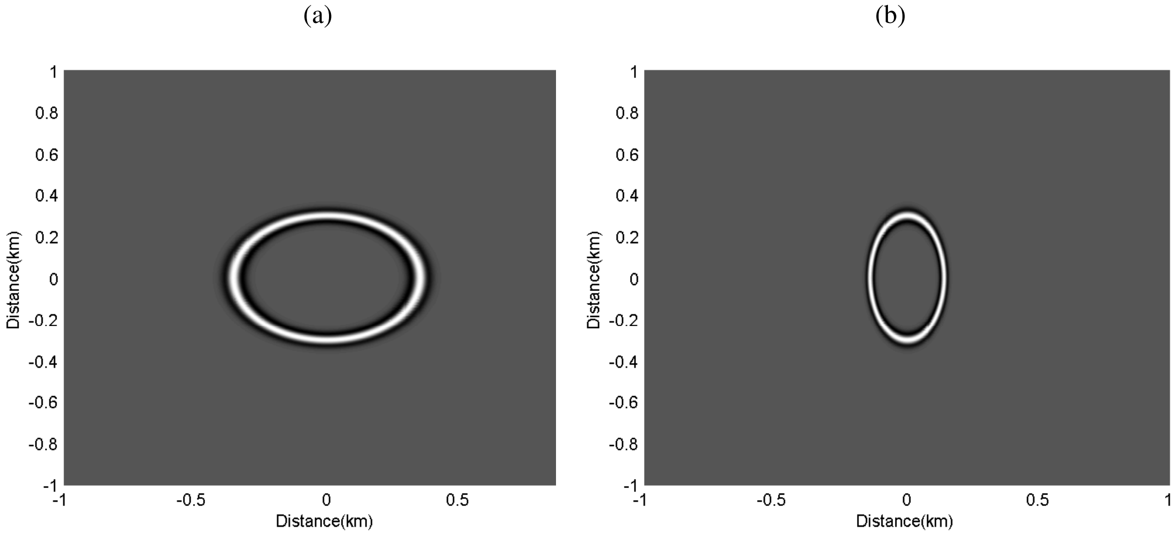

Figure 1: 2D snapshots of potential ψ in the elliptic case, for a point source located at the origin with $\alpha = 1.5km/s$: (a) $\epsilon = \delta = 0.2$; (b) $\epsilon = \delta = -0.39$.

From this analytical solution, it is clear that the wavefront has an elliptical shape for $\epsilon = \delta$ and $|\delta| \ll 1$ (see Figure 1.) Therefore, in elliptical media with weak anisotropy the system of coupled equations reduces to a single partial differential equation. Moreover, no artifacts are present, in agreement with the comments of Helbig (1983), Dellinger and Muir (1988), Tsvankin (1996), Cohen (1996), Alkhalifah (2000), and Amazonas et al. (2010). This results validates the prescription of using a locally elliptical medium around the source position to avoid the excitation of spurious modes when performing numerical modelling using the pseudo-acoustic wave equation (Fei and Liner, 2008; Pestana et al., 2011; Bloot et al., 2011).

WAVE-MODE SEPARATION

There is an important application of the generalized Helmholtz decomposition of vector wavefields in VTI media: the separation of wavefields that travel with different velocities. Such a separation is essential for the application of wave-equation migration, because wavefield components that travel with different velocities cannot be treated correctly at once and will thus lead to cross-talk and other spurious events.

In the case of an isotropic elastic medium with multi-component data, Helmholtz decomposition (7) can be used to separate the original wavefield in other two: a pressure field P and a transverse field S. In this section we show that it is possible to define operators that enable the separation of the q-P and q-S wave fields in VTI media. Other attempts in this directions have previously been made in the literature. Dellinger and Etgen (1990) suggested separating the wavefield in an anisotropic medium by projection of the wavefield in directions where the q-P and q-S waves are polarised. For heterogeneous media Yan and Sava (2009) suggested solving the Christoffel equation using finite-difference approximations to compute wavefield derivatives. In this section, we demonstrate that our generalized Helmholtz decomposition enables us to establish two combined potentials, ϕ_{qP} and ψ_{qSV}, which separately describe the corresponding wave-modes in a VTI medium. These scalar fields can be used in imaging conditions to reduce cross-talk when migrating multicomponent seismic data.

q-P Wave-Mode

According to equation (16), the potential ϕ_{qP} describing the q-P wave mode is given by

$$\phi_{qP} = \nabla \cdot \mathbf{u} + \delta \widehat{\nabla} \cdot \mathbf{u}. \tag{46}$$

Expression (46) indicates that for a homogeneous medium we can separate the q-P wave from the q-SV wave.

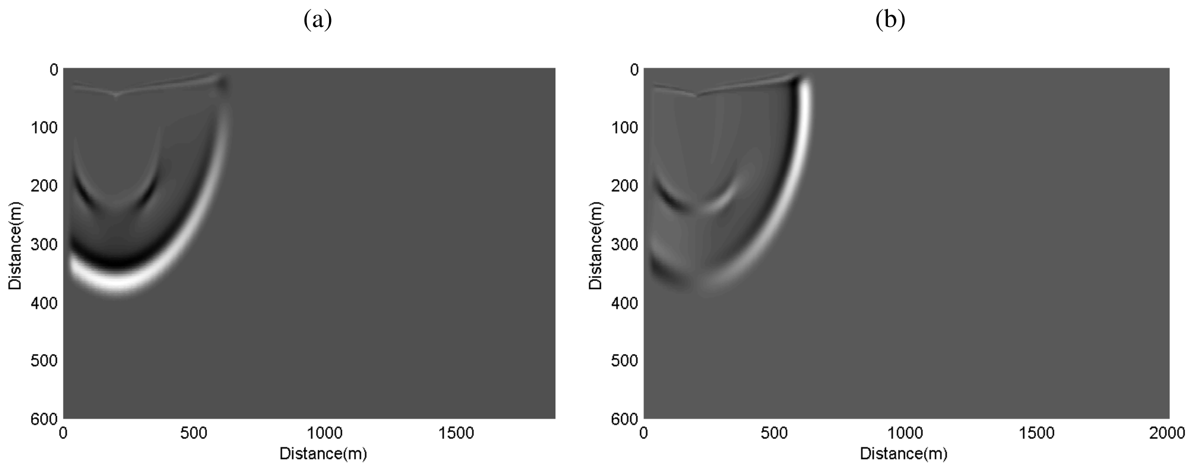

Figure 2: 2D snapshots for (a) u_1 and (b) u_3 in a VTI homogeneous medium with $\epsilon = -0.2187$ and $\delta = -0.2704$. We can observe the coupling between P and S waves.

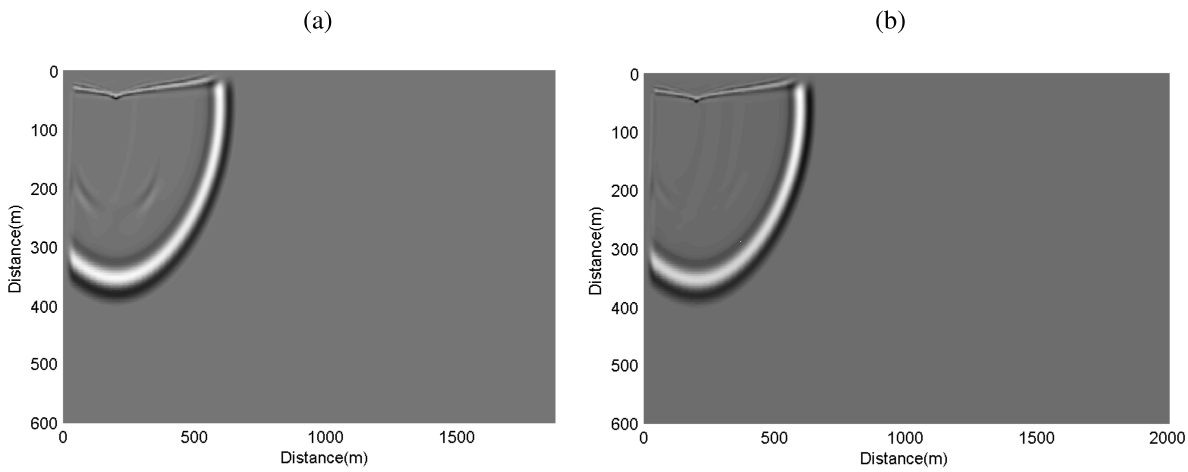

Figure 3: Wavefields after applying the (a) divergence operator and (b) operator (46) to the snapshots u_1 and u_3 in Figure 2. We can observe a better suppression of the S wave for the operator given by (46).

We can see that our method provides a simple way to obtain a pure q-P wavefield potential. Figure 2 shows the snapshots of the horizontal (u_1) and vertical (u_3) components of the wavefield for a homogeneous VTI medium with $\delta = -0.2704$ and $\epsilon = -0.2187$ at some fixed time $t > 0$. Applying operation (46) to components u_1 and u_3 we get the q-P wavefield, which is shown in Figure 3b. Figure 3a shows result after the application of the isotropic operator (divergence) to the snapshots in Figure 2. Equation (46) enables us to separate the q-P wave mode only by knowing δ. It is evident from Figure 2 that there is a coupling between the waves and in the homogeneous case the operator (46) provides only the q-P wave. Our operator achieves a better suppression of the S wave as compared to the isotropic divergence operator $\nabla \cdot \mathbf{u}$.

q-SV wave-mode

Yan and Sava (2009) show in their work, by means of numerical experiments, that the rotational (curl) operator, when applied to u_1 and u_3 cannot completely separate the q-P and q-SV wave-modes. Let us now show that our method offers a better understanding of this fact, as well as a different way, similar to the isotropic classic case, of solving this problem for a homogeneous VTI medium.

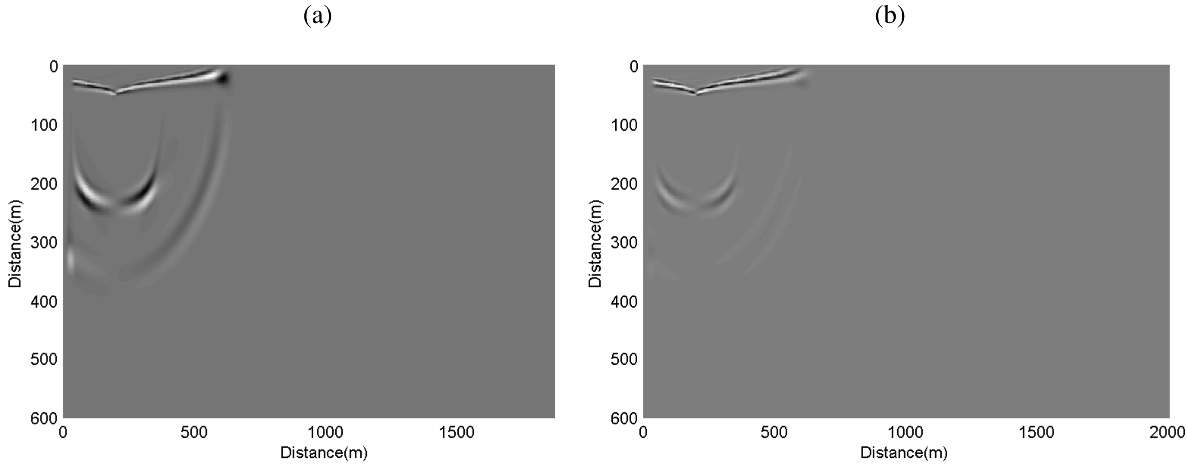

Figure 4: Wavefields after applying the (a) curl operator and (b) operator (52) to the snapshots u_1 and u_3 in Figure 2. The curl operator fails to completely separate P and S wavefields for the direct wave, whereas our provides a much better result.

According to our decomposition, in the 2D case the wavefield $\mathbf{u}$ is given by

$$\mathbf{u} = \nabla\psi + \widehat{\nabla}\phi + \nabla \times \mathbf{\Psi} \quad \text{with} \quad \nabla \cdot \mathbf{\Psi} = 0, \quad \text{and} \quad \widehat{\nabla} \cdot (\nabla \times \mathbf{\Psi}) = 0. \tag{47}$$

Applying the curl, we find

$$\nabla \times \mathbf{u} = \nabla \times \widehat{\nabla}\phi - \Delta\mathbf{\Psi}. \tag{48}$$

From this expression it becomes immediately clear that the curl does not remove the ϕ influence. This explains the results of Yan and Sava (2009) demonstrating that it is unsuccessful to use this operator to separate wavefields in a VTI medium.

However, application of the modified divergence ($\widehat{\nabla}\cdot$) to equation (48) leads to

$$\widehat{\nabla} \cdot (\nabla \times \mathbf{u}) = -\widehat{\nabla} \cdot \Delta\mathbf{\Psi}, \tag{49}$$

where we have used that $\widehat{\nabla} \cdot \left(\nabla \times \widehat{\nabla}\phi\right) = 0$. Therefore, the dependence on ϕ is removed. Now, using equation (17), we find

$$\frac{\partial^2}{\partial t^2}\mathbf{\Psi} = -\beta^2 \nabla \times \mathbf{u}, \tag{50}$$

and thus,

$$\frac{\partial^2}{\partial t^2}\left(\widehat{\nabla} \cdot \mathbf{\Psi}\right) = \beta^2 \Delta \left(\widehat{\nabla} \cdot \mathbf{\Psi}\right). \tag{51}$$

Equation (51) has the same form of equation (28). Thus, we can extract the q-SV wavefield by means of the operator

$$\psi_{qSV} = \widehat{\nabla} \cdot (\nabla \times \mathbf{u}). \tag{52}$$

In Figure 4a we show the application of the above operator to the data depicted in Figure 2. We observe that the q-P wave mode of the direct wave was removed in a better way then by the curl operator (Figure 4b).

CONCLUSIONS

In homogeneous isotropic media, elastic wave propagation can be decomposed into two independent contributions described by separate wave equations. In this work, we have derived an extension of this Helmholtz decomposition for a homogeneous VTI medium. For this purpose, we have represented the elastic wavefield in VTI media by means of generalized potential functions, which help in the construction of analytical solutions to the wave equation for weakly anisotropic VTI media (Bloot et al., 2011). These potentials can be obtained by solving a simpler set of equations, each containing a reduced number of anisotropy parameters.

This generalized Helmholtz decomposition can be applied for the analytical or numerical description of q-P and q-S waves in weakly anisotropic VTI media. In this paper, we have explored some of the consequences of the decomposition. Specifically, we have deduced a new set of two-way pseudo-acoustic wave equation and a new approximation for qP waves. Moreover, we were able to provide a new derivation of the dispersion relations for the one-way pseudo-acoustic wave equation given by Alkhalifah (2000). These results are useful for modelling and imaging of seismic wavefields.

The generalized Helmholtz decomposition leads to a new algorithm for wave-mode separation, which we have tested on a simple synthetic example. Our numerical experiments indicate the effectiveness of the proposed algorithm in a heterogeneous VTI medium. This new approach can be a feasible way to reduce cross-talk between wave modes in elastic imaging.

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APPENDIX A

PROOF OF THE WAVEFIELD SEPARATION THEOREM

Applying the divergence operator to equation (4), we find

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u} = \quad & \alpha^2\Delta\left(\nabla\cdot\mathbf{u}\right)+\alpha^2\Delta_r\left(2(\epsilon-\delta)\widehat{\nabla}\cdot\mathbf{u}\right)+\alpha^2\Delta\left(\delta\widehat{\nabla}\cdot\mathbf{u}\right)\\ & +\alpha^2\Delta_r\left(\delta\widehat{\nabla}\cdot\mathbf{u}\right)+\Delta\left(\frac{\psi_f}{\rho}\right)+\Delta_r\left(\frac{\phi_f}{\rho}\right), \end{aligned} \tag{53}$$

which can be rewritten as

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u}=\Delta\left(\alpha^2\left[\nabla\cdot\mathbf{u}+\delta\widehat{\nabla}\cdot\mathbf{u}+\frac{\psi_f}{\rho\alpha^2}\right]\right)+\Delta_r\left(\alpha^2\left[\delta\nabla\cdot\mathbf{u}+2(\epsilon-\delta)\widehat{\nabla}\cdot\mathbf{u}+\frac{\phi_f}{\rho\alpha^2}\right]\right). \tag{54}$$

Integrating twice in the variable t, we have

$$\begin{aligned}\nabla\cdot\mathbf{u} = \quad & \Delta\left(\int_0^t\int_0^\tau\alpha^2\left[\nabla\cdot\mathbf{u}+\delta\widehat{\nabla}\cdot\mathbf{u}+\frac{\psi_f}{\rho\alpha^2}\right]d\tau' d\tau\right)\\ & +\Delta_r\left(\int_0^t\int_0^\tau\alpha^2\left[\delta\nabla\cdot\mathbf{u}+2(\epsilon-\delta)\widehat{\nabla}\cdot\mathbf{u}+\frac{\phi_f}{\rho\alpha^2}\right]d\tau' d\tau\right), \end{aligned} \tag{55}$$

and using equations (16) and (18), it follows that

$$\nabla\cdot\mathbf{u}=\Delta_r\left(\psi+\phi\right)+\frac{\partial^2}{\partial z^2}\psi. \tag{56}$$

Analogously, applying the operator $\widehat{\nabla}\cdot$ to equation (4) and using condition (21), we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\widehat{\nabla}\cdot\mathbf{u} = & \,\alpha^2\Delta_r\left(\nabla\cdot\mathbf{u}\right)+\alpha^2\Delta_r\left(2(\epsilon-\delta)\delta\widehat{\nabla}\cdot\mathbf{u}\right)+\alpha^2\Delta_r\left(\delta\widehat{\nabla}\cdot\mathbf{u}\right)\\ & +\Delta_r\left(\delta\nabla\cdot\mathbf{u}\right)+\Delta_r\left(\frac{\psi_f+\phi_f}{\rho}\right)+\frac{\widehat{\nabla}\cdot\left(\nabla\times\mathbf{\Psi}_f\right)}{\rho}, \end{aligned} \tag{57}$$

and integrating twice in the variable t again, we obtain

$$\widehat{\nabla}\cdot\mathbf{u}=\alpha^2\Delta_r\left(\psi+\phi\right)+\frac{1}{\rho}\int_0^t\int_0^\tau\widehat{\nabla}\cdot\left(\nabla\times\mathbf{\Psi}_f\right)d\tau' d\tau. \tag{58}$$

With expressions (56) and (58), we can show that ψ and ϕ satisfy equations (26) and (27). For that purpose, we take the second time-derivative of equation (16) to find

$$\frac{\partial^2}{\partial t^2}\psi=\alpha^2\left[(1+\delta)\Delta_r\left(\psi+\phi\right)+\frac{\partial^2}{\partial z^2}\psi\right]+\frac{\psi_f}{\rho}+\frac{\alpha^2\delta}{\rho}\int_0^t\int_0^\tau\widehat{\nabla}\cdot\left(\nabla\times\mathbf{\Psi}_f\right)d\tau' d\tau. \tag{59}$$

In an analogous way, we find from equation (17) that

$$\frac{\partial^2}{\partial t^2}\phi=\alpha^2\left[(2\epsilon-\delta)\Delta_r\left(\psi+\phi\right)+\delta\frac{\partial^2}{\partial z^2}\psi\right]+\frac{\phi_f}{\rho}+\frac{2(\epsilon-\delta)\alpha^2}{\rho}\int_0^t\int_0^\tau\widehat{\nabla}\cdot\left(\nabla\times\mathbf{\Psi}_f\right)d\tau' d\tau. \tag{60}$$

Using condition (14), we can take

$$F_1 = \frac{\psi_f}{\rho} \quad \text{and} \quad F_2 = \frac{\phi_f}{\rho}\,, \tag{61}$$

which demonstrates that ψ and ϕ satisfy equation (26).

Next, we apply the operator $\widehat{\nabla}^{\perp}\cdot$ to equation (4),

$$\frac{\partial^2}{\partial t^2}\widehat{\nabla}^{\perp}\cdot\mathbf{u} = \beta^2\Delta\left(\widehat{\nabla}^{\perp}\cdot\mathbf{u}\right) + 2\gamma\beta^2\Delta_r\left(\widehat{\nabla}^{\perp}\cdot\mathbf{u}\right) + \Delta_r\left(\frac{\Phi_f}{\rho}\right). \tag{62}$$

Multiplying the above equation by 2γ and integrating twice in time, we find

$$\begin{aligned} 2\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u} \quad = \quad & \beta^2\Delta\left(\beta^2\int_0^t\int_0^\tau\left[\left(2\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}\right) + \frac{1}{\rho\beta^2}\Phi_f\right]d\tau' d\tau\right) \\ & + 2\gamma\beta^2\Delta_r\left(\beta^2\int_0^t\int_0^\tau\left[\left(2\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}\right) + \frac{1}{\rho\beta^2}\Phi_f\right]d\tau' d\tau\right) - \Delta\left(\frac{\Phi_f}{\rho\beta^2}\right), \end{aligned} \tag{63}$$

which allows us to conclude that

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}\Phi = (1+2\gamma)\Delta_r\Phi + \frac{\partial^2}{\partial z^2}\Phi + \frac{1}{\rho\beta^2}\Phi_f - \Delta\left(\frac{\Phi_f}{\rho\beta^2}\right), \tag{64}$$

which is equation (27) with the choice

$$F_3 = \frac{1}{\rho\beta^2}\Phi_f - \Delta\left(\frac{\Phi_f}{\rho\beta^2}\right). \tag{65}$$

To confirm equation (28) we must apply the curl operator to equation (4) and integrate twice in t. Then, we have that expression (17) satisfies equation (27). Moreover, condition (29) follows immediately from condition (21).

APPENDIX B

THE ONE-WAY SOLUTION

System (36)–(37) can be written as

$$\begin{aligned} \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\varphi - (1+2\epsilon)\frac{\partial^2\varphi}{\partial x^2} &= (1+\delta)\frac{\partial^2\psi}{\partial z^2} \\ \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2\psi}{\partial z^2} &= (1+\delta)\frac{\partial^2\varphi}{\partial x^2}. \end{aligned} \tag{66}$$

Applying the two-dimensional Fourier transform in time and space to the above equations, we obtain

$$\widehat{\varphi} = \left[\frac{-(1+\delta)}{\omega^2/\alpha^2 - (1+2\epsilon)k_x^2}\right]\frac{\partial^2\widehat{\psi}}{\partial z^2} \tag{67}$$

and

$$\frac{\omega^2}{\alpha^2}\widehat{\psi} + \frac{\partial^2\widehat{\psi}}{\partial z^2} = (1+\delta)k_x^2\widehat{\varphi}. \tag{68}$$

Substitution of equation (67) in (68) leads to

$$\frac{\omega^2}{\alpha^2}\widehat{\psi} + \frac{\partial^2\widehat{\psi}}{\partial z^2} = \left[\frac{-(1+\delta)^2k_x^2}{\frac{\omega^2}{\alpha^2} - (1+2\epsilon)k_x^2}\right]\frac{\partial^2\widehat{\psi}}{\partial z^2}. \tag{69}$$

Since we are assuming $|\delta| \ll 1$, the following expression is valid

$$(1+\delta)^2 = 1 + 2\delta + \delta^2 \approx 1 + 2\delta. \tag{70}$$

Thus, expression (69) reduces to

$$\frac{\omega^2}{\alpha^2}\widehat{\psi} + \left[1 + \frac{(1+2\delta)k_x^2}{\frac{\omega^2}{\alpha^2} - (1+2\epsilon)k_x^2}\right]\frac{\partial^2\widehat{\psi}}{\partial z^2} = 0. \tag{71}$$

After some algebraic manipulations, we have

$$\frac{\partial^2\widehat{\psi}}{\partial z^2} + \frac{\omega^2}{\alpha^2}\left[\frac{\frac{\omega^2}{\alpha^2} - (1+2\epsilon)k_x^2}{\frac{\omega^2}{\alpha^2} - 2(\epsilon-\delta)k_x^2}\right]\widehat{\psi} = 0. \tag{72}$$

Defining the anisotropic parameter η in accordance with Alkhalifah (2000) as

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta}, \tag{73}$$

we can write

$$\frac{\partial^2\widehat{\psi}}{\partial z^2} + k_z^2\widehat{\psi} = 0, \tag{74}$$

where

$$k_z^2 = \frac{\omega^2}{\alpha^2}\left[\frac{\frac{\omega^2}{\alpha^2} - (1+2\delta)(1+2\eta)k_x^2}{\frac{\omega^2}{\alpha^2} - 2(1+2\delta)\eta k_x^2}\right]. \tag{75}$$

From the definition of the NMO velocity (Tsvankin, 2001),

$$V_{NMO} = \alpha\sqrt{1+2\delta}, \tag{76}$$

we obtain

$$k_z = \pm\frac{\omega}{\alpha}\sqrt{1 - \frac{\frac{V_{NMO}^2 k_x^2}{\omega^2}}{1 - 2\eta\frac{V_{NMO}^2 k_x^2}{\omega^2}}}. \tag{77}$$

Finally, using the notation

$$m^2 = \frac{V_{NMO}^2 k_x^2}{\omega^2}, \tag{78}$$

introduced by Fei and Liner (2008), we arrive at

$$k_z = \pm\frac{\omega}{\alpha}\sqrt{1 - \frac{m^2}{1 - 2\eta m^2}}. \tag{79}$$