ON THE ELASTIC WAVE EQUATION FOR WEAKLY ANISOTROPIC VTI MEDIA

R. Bloot, J. Schleicher, and L. T. Santos

email: *rodrigo.bloot@unila.edu.br, js@ime.unicamp.br*
keywords: *Wave equation, weak anisotropy, VTI medium*

ABSTRACT

We derive a general elastic wave equation in weakly anisotropic VTI media by linearizing the expression of the stiffness tensor in terms of the Thomsen parameters. The resulting wave equation is a system of three coupled differential equations for the three components of the displacement vector. For $\delta = 0$, the third equation becomes an independent equation for the third component of the particle displacement, identical to the isotropic situation, and the first two equations remain coupled. Using zero-order ray theory, we derive the associated eikonal and transport equations for q-P, q-SV and q-SH waves. These are finally reduced to the pseudo-acoustic case where the vertical S-wave velocity is zero. This allows for a better understanding of the pseudo-S wave artifact in such media.

INTRODUCTION

The knowledge of the wave equation is of fundamental importance for a good and satisfying understanding of the phenomena of wave propagation. Wave propagation in anisotropic media has been studied by many authors. One reason for this is that the solution of the elastic wave equation numerically simulates the behaviour of the wave in a given environment. In an isotropic medium the wave field is composed of two types of waves: Compressional or primary (P) waves and shear or secondary (S) waves. In an anisotropic medium, this wave field is composed of three types of waves. Because of the similarity of their polarization to the isotropic case, these are called quasi-P, quasi-SV and quasi-SH waves. The separation of the two types of S waves is known as birefringence. For more details, the reader is referred to the book of Červený (2001) and Tsvankin (2001).

However, it is unsatisfactory to work with the full anisotropic wave equation in media with certain symmetries. Of particular interest is the case of media with transversal isotropy and a vertical symmetry axis, also shortly referred to as VTI media. Several attempts have been undertaken over the past decade to describe wave propagation in VTI media using the medium symmetry to simplify the governing equations (see, e.g., Tsvankin, 1996; Pestana et al., 2011). Many authors have developed approximations to simplify the anisotropic equations, using weak anisotropy approximations (Thomsen, 1986; Cohen, 1996), the elliptical approximation (Helbig, 1983; Dellinger and Muir, 1988), small dip-angle approximations (Cohen, 1996), to cite a few.

One very helpful approximation for the description of wave propagation in VTI media is the linearization of Thomsen (1986) for weak anisotropy. In this work we use this linearization directly in the stiffness tensor which relates the material properties of a VTI medium to wave propagation. In this form, we derive a form of the tensor that allows to describe the wave equation in a much more elegant, compact and functional way than before in the literature. Armed with this equation, its possible to apply, modify, and generalize the known techniques of high-frequency approximation for solving PDE's.

Using a zero-order ray approximation in the elastic equation, we study the three wave modes independently, deriving the corresponding eikonal and transport equations. One particular advantage of the new derivation is the more natural treatment of the pseudo-acoustic case (where the vertical S-wave velocity is assumed to be zero). Alkhalifah (2000) derived the equation for a pseudo-acoustic VTI medium using dispersion relations. Besides him, Duveneck et al. (2008), Liu and Sen (2010) and recently Pestana et al. (2011) worked on the acoustic case. In our approach, the equations for the pseudo-acoustic VTI medium are obtained as a special case of the general elastic equations.

WAVE EQUATION IN A VTI MEDIUM

In this section we derive the elastic wave equation for a weakly anisotropic VTI medium. We start from the wave equation for general anisotropic heterogeneous media (Aki and Richards, 1980)

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = f_i. \tag{1}$$

Here, $\mathbf{u} = (u_1, u_2, u_3)^T$ is the displacement vector, $\mathbf{f} = (f_1, f_2, f_3)^T$ is the (external) body force, t is time, and $x_1 = x$, $x_2 = y$, and $x_3 = z$ denote Cartesian coordinates. Einstein's convention will be used whenever convenient.

The general elastic stiffness tensor, c_{ijkl}, has several symmetries that reduce the number of independent elements. Because of the symmetry of the stress and strain tensors involved, we have

$$c_{ijkl} = c_{jikl} \quad \text{and} \quad c_{ijkl} = c_{ijlk}. \tag{2}$$

Moreover, due to thermodynamic considerations we have also

$$c_{ijkl} = c_{klij}. \tag{3}$$

These symmetries reduce the number of independent elements fro $3^4 = 81$ to 21 in a triclinic medium. Using these symmetries and the compact Voigt notation ($ij \to \alpha$ and $kl \to \beta$, with $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$, and $12 \to 6$) we can represent the stiffness tensor as a 6×6 matrix $C_{\alpha\beta}$.

VTI medium

Additional medium symmetries further reduce the number of independent elements. Each anisotropic symmetry is characterized by a specific structure of the matrix $C_{\alpha\beta}$. For a VTI medium, it reduces to

$$\mathbf{C}^{\mathrm{VTI}} = \begin{bmatrix} C_{11} & C_{11}-2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11}-2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{4}$$

For an isotropic medium, equation (4) further reduces to

$$\mathbf{C}^{\mathrm{ISO}} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}, \tag{5}$$

where λ and μ are the Lamé parameters.

It is useful to express matrix (4) in terms of the anisotropy parameters of Thomsen (1986), which are given by

$$\epsilon = \frac{C_{11}-C_{33}}{2C_{33}}, \tag{6}$$

$$\delta = \frac{(C_{13}+C_{55})^2-(C_{33}-C_{55})^2}{2C_{33}(C_{33}-C_{55})}, \tag{7}$$

$$\gamma = \frac{C_{66}-C_{55}}{2C_{55}}. \tag{8}$$

Comparing equations (4) and (5), we find it convenient to express the parameters C_{33} and C_{55} for a VTI medium in terms of generalized Lamé parameters as

$$C_{33} = \lambda + 2\mu, \quad \text{and} \quad C_{55} = \mu. \tag{9}$$

Using equations (6) and (9), we can rewrite parameter C_{11} as

$$C_{11} = C_{33}(2\epsilon + 1) = (\lambda + 2\mu)(2\epsilon + 1), \tag{10}$$

and from equations (8) and (9), we obtain for C_{66}

$$C_{66} = C_{55}(2\gamma + 1) = \mu(2\gamma + 1). \tag{11}$$

Moreover, upon the use of equations (9), from equation (7) we have

$$C_{13} = -C_{55} \pm (C_{33} - C_{55})\sqrt{1 + \frac{2C_{33}\delta}{C_{33} - C_{55}}} = -\mu \pm (\lambda + \mu)\sqrt{1 + \frac{2(\lambda + 2\mu)\delta}{\lambda + \mu}}. \tag{12}$$

The two possible signs in equation (12) reflect the fact that C_{13} can only be uniquely determined from δ if the sign of $C_{13} + C_{44}$ is known (Tsvankin, 1996).

Weak anisotropy

For weak anisotropy ($|\delta| \ll 1$) this expression can be approximated to first order as

$$C_{13} = -\mu \pm (\lambda + \mu)\left[1 + \frac{(\lambda + 2\mu)\delta}{\lambda + \mu}\right]. \tag{13}$$

The sign to be chosen in equation (13) is the positive one, as explained next. The P and S wave velocities along the symmetry axis are given by

$$\alpha = \sqrt{\frac{C_{33}}{\rho}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \text{and} \quad \beta = \sqrt{\frac{C_{55}}{\rho}} = \sqrt{\frac{\mu}{\rho}}, \tag{14}$$

respectively. Since physical considerations require that $\alpha > \beta$, we can conclude that $C_{33} - C_{55} = \lambda + \mu > 0$. Moreover, we know (Tsvankin, 2001) that a lower bound for δ is given by

$$\delta_{min} = -\frac{1}{2}\left(1 - \frac{\alpha^2}{\beta^2}\right). \tag{15}$$

The condition $\delta > \delta_{min}$ is equivalent to

$$\frac{2\beta^2\delta}{\alpha^2 - \beta^2} + 1 > 0, \tag{16}$$

and then,

$$\frac{2C_{33}\delta}{C_{33} - C_{55}} = \frac{2\alpha^2\delta}{\alpha^2 - \beta^2} > \frac{2\beta^2\delta}{\alpha^2 - \beta^2}. \tag{17}$$

Therefore, the expression under the square root in equation (12) is positive. Because equation (13) has to tend to $C_{13} = \lambda$ for $\delta \to 0$, we conclude that in first-order approximation,

$$C_{13} = \lambda + \delta(\lambda + 2\mu). \tag{18}$$

This is the expression for C_{13} that describes a weakly anisotropic VTI medium (Thomsen, 1993). We stress that the condition $|\delta| \ll 1$ is sufficient for the linearization of equation (12). No further assumptions are needed about ϵ and γ.

Substituting of the above expressions back in equation (4), we find the weak-anisotropy approximation for the stiffness matrix of a VTI medium to be

$$\mathbf{C}_{\mathrm{weak}}^{\mathrm{VTI}} = \begin{bmatrix} (\lambda+2\mu)(1+2\epsilon) & \lambda(1+2\epsilon)+4\mu(\epsilon-\gamma) & \lambda+\delta(\lambda+2\mu) & 0 & 0 & 0 \\ \lambda(1+2\epsilon)+4\mu(\epsilon-\gamma) & (\lambda+2\mu)(1+2\epsilon) & \lambda+\delta(\lambda+2\mu) & 0 & 0 & 0 \\ \lambda+\delta(\lambda+2\mu) & \lambda+\delta(\lambda+2\mu) & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu(1+2\gamma) \end{bmatrix}. \tag{19}$$

Substituting this expression for the stiffness tensor back in equation (1) results in the elastic wave equation for weakly anisotropic VTI media (see Appendix A),

$$\begin{aligned} \rho\frac{\partial^2}{\partial t^2}\mathbf{u} = \mathbf{f} \quad &+ \quad \nabla\cdot[\mu\nabla\mathbf{u}] + \nabla[(\lambda+\mu)\nabla\cdot\mathbf{u}] + \nabla\mu\times(\nabla\times\mathbf{u}) + 2(\nabla\mu\cdot\nabla)\mathbf{u} \\ &+ \quad \widehat{\nabla}[\delta(\lambda+2\mu)\nabla\cdot\mathbf{u}] + \nabla[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}] - 2\widehat{\nabla}[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}] \\ &+ \quad 2\widehat{\nabla}^{\perp}[\mu\gamma\widehat{\nabla}^{\perp}\cdot\mathbf{u}] + 4\mathbf{J}\nabla\mathbf{u}\widehat{\nabla}^{\perp}[\mu\gamma] + 2\widehat{\nabla}[\epsilon(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}]. \end{aligned} \tag{20}$$

Here, we have used the notations

$$\widehat{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\right)^T \tag{21}$$

and

$$\widehat{\nabla}^{\perp} = \mathbf{J}\widehat{\nabla} = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}, 0\right)^T, \tag{22}$$

with

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Moreover, the (i,j)-element of the Jacobian matrix $\nabla\mathbf{u}$ is $\partial u_i/\partial x_j$. Equation (20) describes a system of three coupled differential equations, because all components of $\mathbf{u}$ appear in all three equations of system (20). For $\delta = 0$, the third equation becomes an independent equation for u_3, identical to the isotropic situation, and the first two equations remain coupled for u_1 and u_2. It is easy to see that equation (20) reduces to the isotropic case when $\delta = \gamma = \epsilon = 0$.

RAY-THEORETICAL SOLUTION

Our next step is to find approximate solutions to equation (20) using a high-frequency asymptotic approximation to obtain information about the kinematics and dynamics of the problem. Using zero-order ray theory we can see that expression (19) is sufficient to obtain an approximation of the solution of the wave equation for this kind of media (Červený, 2001; Pujol, 2003).

We start from equation (1) without a source term, i.e.,

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right). \tag{24}$$

According to zero-order ray theory (Červený, 1985; Červený, 2001), approximate solutions take the form

$$\mathbf{u}(x,y,z,t) = \mathbf{U}(x,y,z)g(t-T(x,y,z)), \tag{25}$$

or, in the frequency domain,

$$\check{\mathbf{u}}(x,y,z,\omega) = \mathbf{U}(x,y,z)\check{g}(\omega)e^{-i\omega T(x,y,z)}, \tag{26}$$

where g is a convenient function describing the source wavelet. Substituting expression (25) in equation (24) we obtain an eigenvalue problem

$$\mathbf{\Gamma}\mathbf{U} = \mathbf{U}, \tag{27}$$

where the Christoffel matrix $\mathbf{\Gamma}$ has elements

$$\Gamma_{ik} = \frac{1}{\rho} c_{ijkl} \frac{\partial T}{\partial x_j} \frac{\partial T}{\partial x_l}. \tag{28}$$

Note that the kinematics of the problem depends only on the stiffness tensor c_{ijkl} and not on its derivatives.

Eikonal equations

We now substitute expression (19) in expression (28) in order to obtain the Christoffel matrix. After introduction of the slowness vector $\mathbf{p}$ and the horizontal slowness vector $\widehat{\mathbf{p}}$, defined as

$$\mathbf{p} = \nabla T, \quad \text{and} \quad \widehat{\mathbf{p}} = \widehat{\nabla} T. \tag{29}$$

we can write the elements of the Christoffel matrix as

$$\begin{aligned}
\Gamma_{11} &= (\alpha^2 - \beta^2)p_1^2 + \beta^2 \|\mathbf{p}\|^2 + 2\epsilon\alpha^2 p_1^2 + 2\gamma\beta^2 p_2^2, \\
\Gamma_{22} &= (\alpha^2 - \beta^2)p_2^2 + \beta^2 \|\mathbf{p}\|^2 + 2\epsilon\alpha^2 p_2^2 + 2\gamma\beta^2 p_1^2, \\
\Gamma_{33} &= (\alpha^2 - \beta^2)p_3^2 + \beta^2 \|\mathbf{p}\|^2, \\
\Gamma_{21} &= \Gamma_{12} = (\alpha^2 - \beta^2)p_1 p_2 + 2(\epsilon\alpha^2 - \gamma\beta^2)p_1 p_2, \\
\Gamma_{31} &= \Gamma_{13} = (\alpha^2 - \beta^2)p_1 p_3 + \delta\alpha^2 p_1 p_3, \\
\Gamma_{32} &= \Gamma_{23} = (\alpha^2 - \beta^2)p_2 p_3 + \delta\alpha^2 p_2 p_3,
\end{aligned} \tag{30}$$

where we have used equations (14). As before, note that $\mathbf{\Gamma}$ reduces to the isotropic case when $\epsilon = \delta = \gamma = 0$.

Exact eigenvalues. Since we have so far only linearized in δ, but allowed for arbitrary ϵ and γ, it is interesting to calculate the three eigenvalues (Λ) of $\mathbf{\Gamma}$ exactly. They are given by

$$\Lambda_{1,2} = \frac{1}{2}\left((\alpha^2 + \beta^2)\|\mathbf{p}\|^2 + 2\epsilon\alpha^2\|\widehat{\mathbf{p}}\|^2 \pm \sqrt{(\alpha^2 - \beta^2)^2\|\mathbf{p}\|^4 + 4\Pi} \right), \tag{31}$$

and

$$\Lambda_3 = \beta^2\|\mathbf{p}\|^2 + 2\gamma\beta^2\|\widehat{\mathbf{p}}\|^2, \tag{32}$$

where

$$\Pi = \alpha^2 \left[(\alpha^2 - \beta^2)\left(\epsilon\|\mathbf{p}\|^2 + 2(\delta - \epsilon)p_3^2\right) + \alpha^2\left(\epsilon^2\|\widehat{\mathbf{p}}\|^2 + \delta^2 p_3^2\right) \right] \|\widehat{\mathbf{p}}\|^2. \tag{33}$$

Equation (27) requires that the eigenvalues must be equal to one, which gives rise to three different eikonal equations. They are given by

$$\alpha^2\|\mathbf{p}\|^2 + \epsilon\alpha^2\|\widehat{\mathbf{p}}\|^2 + \Theta = 1, \tag{34}$$

$$\beta^2\|\mathbf{p}\|^2 + \epsilon\alpha^2\|\widehat{\mathbf{p}}\|^2 - \Theta = 1, \tag{35}$$

and

$$\beta^2\|\mathbf{p}\|^2 + 2\gamma\beta^2\|\widehat{\mathbf{p}}\|^2 = 1, \tag{36}$$

where

$$\Theta = \frac{1}{2}(\alpha^2 - \beta^2)\|\mathbf{p}\|^2 \left[\sqrt{1 + \frac{4\Pi}{(\alpha^2 - \beta^2)^2\|\mathbf{p}\|^4}} - 1 \right]. \tag{37}$$

We recognize that equations (34) and (35) do not depend on γ, as expected (Tsvankin, 1996).

Equations (34) to (36) define the traveltime of q-P, q-SV and q-SH waves in heterogeneous VTI media. Substitution of $\|\mathbf{p}\|^2 = 1/V^2$, $p_3^2 = \cos^2\theta/V^2$, and $\|\widehat{\mathbf{p}}\|^2 = \sin^2\theta/V^2$, where θ is the phase angle and V is the phase velocity, yields the corresponding dispersion relations for the phase velocities. Note that these equations are valid for small δ but arbitrary ϵ and γ. In the isotropic case, where $\epsilon = \delta = \gamma = 0$, we have

$\Pi = \Theta = 0$. Then, equations (35) and (36) coincide. From equation (36), we immediately recognize the well-known fact that the slowness surface of the q-SH wave is an ellipsoid, because we can rewrite it as

$$v_h^2\|\widehat{\mathbf{p}}\|^2 + v_v^2 p_3^2 = 1, \tag{38}$$

where $v_v = \beta$ is the vertical velocity and $v_h = \beta\sqrt{1+2\gamma}$ is the horizontal velocity.

Since Θ depends on the derivatives of the traveltime, equations (34) and (35) are harder to interpret. First, let us observe that all the equations (34) to (36) can be written in a common notation as

$$v^2\|\mathbf{p}\|^2 = 1 - \kappa, \tag{39}$$

where v represents the velocity of each wave mode along the axis of symmetry and κ is the anisotropy correction, given by

$$\text{q-P:} \qquad \kappa = \epsilon\alpha^2\|\widehat{\mathbf{p}}\|^2 + \Theta, \tag{40}$$

$$\text{q-SV:} \qquad \kappa = \epsilon\alpha^2\|\widehat{\mathbf{p}}\|^2 - \Theta, \tag{41}$$

$$\text{q-SH:} \qquad \kappa = 2\gamma\beta^2\|\widehat{\mathbf{p}}\|^2. \tag{42}$$

In other words, the eikonal equations in a VTI medium can be interpreted as the eikonal equation for the isotropic case modified by an anisotropy correction term. Note again that for the derivation of eikonal equations (39), only $|\delta| \ll 1$ had to be assumed, being valid for arbitrary ϵ and γ.

Weak anisotropy. In this section we approximate the eikonal equations (39) for a weakly anisotropic VTI medium. For this purpose, we impose, additionally to $|\delta| \ll 1$, the condition $|\epsilon| \ll 1$. Linearization of equation (37) in ϵ and δ yields

$$\Theta \approx \left(\epsilon + 2(\delta - \epsilon)\frac{p_3^2}{\|\mathbf{p}\|^2}\right)\alpha^2\|\widehat{\mathbf{p}}\|^2. \tag{43}$$

The traveltime derivatives in equation (43) still depend on the Thomsen parameters according to equations (39). Inspection of equations (40) and (41) together with (43) reveals that in both cases, κ has no zero-order term in the Thomsen parameters. Therefore, according to equations (39), in the term multiplying $(\delta - \epsilon)$, we can use the zero-order approximation $\|\mathbf{p}\| \approx 1/v$. This yields the following linear approximations for κ in equations (40) and (41):

$$\text{q-P:} \qquad \kappa \approx 2\left(\epsilon + (\delta - \epsilon)\alpha^2 p_3^2\right)\alpha^2\|\widehat{\mathbf{p}}\|^2, \tag{44}$$

$$\text{q-SV:} \qquad \kappa \approx -2(\delta - \epsilon)\alpha^4 p_3^2\|\widehat{\mathbf{p}}\|^2. \tag{45}$$

Actually, also p_3 still depends on ϵ and δ. Since it multiplies $(\delta - \epsilon)$, it can be approximated to zeroth order by $p_3^2 \approx 1/v^2 - \|\widehat{\mathbf{p}}\|^2$, so that equations (44) and (45) further reduce to

$$\text{q-P:} \qquad \kappa \approx 2\left(\delta + (\epsilon - \delta)\alpha^2\|\widehat{\mathbf{p}}\|^2\right)\alpha^2\|\widehat{\mathbf{p}}\|^2, \tag{46}$$

$$\text{q-SV:} \qquad \kappa \approx 2\left[(\epsilon - \delta)\frac{\alpha^2}{\beta^2} - (\epsilon - \delta)\alpha^2\|\widehat{\mathbf{p}}\|^2\right)\right]\alpha^2\|\widehat{\mathbf{p}}\|^2. \tag{47}$$

These approximations for the anisotropy correction to the eikonal equations in VTI media confirm the well-known fact that the q-P slowness surfaces are ellipsoids only in elliptically anisotropic media, where $\epsilon = \delta$. In that case, both q-P eikonal equation is given by equation (38), where $v_h = \alpha\sqrt{1+2\delta}$ and $v_v = \alpha$. For q-SV waves in the elliptic case, $\kappa \approx 0$, i.e., the eikonal equation reduces to the isotropic one, $\|\mathbf{p}\|^2 = 1/\beta^2$. For general VTI media with $\epsilon \neq \delta$, the slowness surfaces for q-P and q-SV waves are quartic in the x and y coordinates, and the vertical velocities remain unchanged.

For each eikonal we can associate the corresponding transport equation, and thus obtain the amplitude information for each wave mode. Therefore, it is possible to find an approximate solution of the wave equation in a heterogeneous VTI medium.

SOLUTION OF THE LINEARIZED EIKONAL EQUATIONS

As before, we can write the eikonal equations (39) with κ given by (42), (44), and (45) in common notation as

$$v^2\Big(\|\mathbf{p}\|^2 + 2\xi\|\widehat{\mathbf{p}}\|^2 + 2\zeta\alpha^2 p_3^2\|\widehat{\mathbf{p}}\|^2\Big) = 1, \tag{48}$$

where v represents the wave mode velocity of the q-P or q-SV waves and the parameters ξ and ζ describe the anisotropy. They are given by

$$\text{q-P}: \quad v = \alpha, \quad \xi = \epsilon, \quad \zeta = \delta - \epsilon, \tag{49}$$

$$\text{q-SV}: \quad v = \beta, \quad \xi = 0, \quad \zeta = \epsilon - \delta, \tag{50}$$

$$\text{q-SH}: \quad v = \beta, \quad \xi = \gamma, \quad \zeta = 0. \tag{51}$$

Once we have found a compact way to represent the eikonal equations for q-P, q-SV and q-SH waves, let us solve equation (48) using the method of characteristics (ray theory).

Equation (48) can be rewritten as

$$H(\mathbf{x}, T, \mathbf{p}) = 0, \tag{52}$$

with

$$H(\mathbf{x}, T, \mathbf{p}) = \frac{v^2}{2}\Big[\|\mathbf{p}\|^2 + 2\xi\|\widehat{\mathbf{p}}\|^2 + 2\zeta\alpha^2 p_3^2\|\widehat{\mathbf{p}}\|^2\Big] - \frac{1}{2}. \tag{53}$$

The method of characteristics then establishes the following system of ordinary differential equations

$$\frac{d\mathbf{x}}{d\tau} = \frac{\partial H}{\partial \mathbf{p}} = v^2\left[\mathbf{p} + 2\xi\widehat{\mathbf{p}} + 2\zeta\alpha^2 p_3\begin{pmatrix} p_1p_3 \\ p_2p_3 \\ \|\widehat{\mathbf{p}}\|^2 \end{pmatrix}\right], \tag{54}$$

$$\frac{d\mathbf{p}}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}} = -v\nabla v\|\mathbf{p}\|^2 - \nabla(\xi v^2)\|\widehat{\mathbf{p}}\|^2 - \nabla(\zeta v^2\alpha^2)p_3^2\|\widehat{\mathbf{p}}\|^2, \tag{55}$$

and

$$\frac{dT}{d\tau} = p_i\frac{\partial H}{\partial p_i} = v^2\Big[\|\mathbf{p}\|^2 + 2\xi\|\widehat{\mathbf{p}}\|^2 + 4\zeta\alpha^2 p_3^2\|\widehat{\mathbf{p}}\|^2\Big], \tag{56}$$

where τ is the traveltime along the ray. These equations can be solved numerically using appropriate initial conditions.

TRANSPORT EQUATIONS

The transport equation for the q-P, q-SH and q-SV waves is (Pujol, 2003)

$$\frac{\partial c_{ijkl}}{\partial x_j}U_k p_l + c_{ijkl}\frac{\partial U_k}{\partial x_j}p_l + c_{ijkl}U_k\frac{\partial p_l}{\partial x_j} + c_{ijkl}\frac{\partial U_k}{\partial x_l}p_j = 0, \tag{57}$$

which can be written in a more compact way as

$$\frac{\partial}{\partial x_j}\Big(c_{ijkl}U_iU_kp_l\Big) = 0. \tag{58}$$

Using approximation (19) in equation (58) will lead us to the general transport equation in a VTI medium with weak anisotropy.

Energy conservation

Using the transport equation (58), it is possible to obtain the law of conservation of energy for weakly anisotropic VTI media (see Appendix B):

$$\nabla\cdot\Big(\rho v^2\|\mathbf{U}\|^2[\mathbf{I} + 2\boldsymbol{\Phi}]\mathbf{p}\Big) = 0, \tag{59}$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{\Phi}$ is the anisotropy correction matrix given by

$$\mathbf{\Phi} = \begin{pmatrix} \xi + \zeta\alpha^2 p_3^2 & 0 & 0 \\ 0 & \xi + \zeta\alpha^2 p_3^2 & 0 \\ 0 & 0 & \zeta\alpha^2 \|\widehat{\mathbf{p}}\|^2 \end{pmatrix}. \tag{60}$$

This is the general transport equation and to solve it we can use the classical ray Jacobian method as in the isotropic case. Let Σ be a closed smooth surface and $\mathbf{n}$ its normal vector. From Gauss' theorem and equation (59) we have,

$$\iint_{\Sigma} \rho v^2 \|\mathbf{U}\|^2 [\mathbf{I} + 2\mathbf{\Phi}]\mathbf{p} \cdot \mathbf{n} \, d\Sigma = 0. \tag{61}$$

We then conclude that if the amplitude $\|\mathbf{U}\|$ is known at a given point, then it can be determined at any other point after computation of the corresponding cross-sectional areas in a manner analogous to the isotropic case.

PSEUDO-ACOUSTIC CASE

In isotropic media ($\epsilon = \delta = \gamma = 0$) in the acoustic case ($\mu = 0$ and $\beta = 0$), there is no S-wave propagation, since the corresponding eigenvalues $\Lambda_{2,3}$ in equations (31) and (32) are zero. For many purposes, it is often advantageous to study the analogous, so-called pseudo-acoustic case for VTI media (Alkhalifah, 2000; Duveneck et al., 2008), where $\beta = 0$, but the Thomsen parameters are not.

Alkhalifah (2000) derived eikonal and wave equations for a pseudo-acoustic VTI medium from the dispersion relation for q-P wave propagation. Duveneck et al. (2008) substituted the pseudo-acoustic condition $c_{55} = 0$ directly in equation (4) and derived a system of coupled equations from there. Here, we have started by linearizing equation (4) for weak anisotropy before considering the pseudo-acoustic case. It is not difficult to verify that the linearized matrix (19) with $\mu = 0$ corresponds to linearization of matrix (4) in the pseudo-acoustic case described by Duveneck et al. (2008).

We start from the exact representations (31) and (32) of the eigenvalues of the Christoffel matrix (30) for the linearized stiffness matrix (19). For $\beta = 0$ and after linearization in δ, they read

$$\Lambda_{1,2} = \frac{1}{2}(a \pm \sqrt{a^2 + b}), \quad \text{and} \quad \Lambda_3 = 0, \tag{62}$$

where

$$a = \alpha^2 \|\mathbf{p}\|^2 + 2\epsilon\alpha^2 \|\widehat{\mathbf{p}}\|^2, \quad \text{and} \quad b = 8\alpha^4(\delta - \epsilon) p_3^2 \|\widehat{\mathbf{p}}\|^2. \tag{63}$$

Note that equations (62) and (63) do not depend on γ anymore, though no assumption has been made about the value of γ. As mentioned before, the eigenvalues (62) must be equal to one to describe the propagation of q-P, q-SV, and q-SH waves in a pseudo-acoustic VTI medium. Let us now analyse these waves in more detail.

Propagation of q-P waves

Let us start with the description of q-P waves in a pseudo-acoustic VTI medium. The condition $\Lambda_1 = 1$ can only be satisfied for real a and b if and only if $a \leq 2$ and

$$a + \frac{b}{4} = 1. \tag{64}$$

Substitution of a and b from equations (63) yields the eikonal equation for the q-P wave in a pseudo-acoustic VTI medium,

$$(1 + 2\epsilon)\alpha^2 \|\widehat{\mathbf{p}}\|^2 + \Big(1 - 2\alpha^2(\epsilon - \delta)\alpha^2 \|\widehat{\mathbf{p}}\|^2\Big)\alpha^2 p_3^2 = 1, \tag{65}$$

Note that though no linearization in ϵ was done, equation (65) is already linear not only in δ, but also in ϵ. In other words, equation (65) is valid for arbitrary values of ϵ. The same equation is obtained when substituting the linearized expression (44) with $\beta = 0$ in equation (39).

In terms of the NMO velocity for a horizontal reflector,

$$V_{NMO} = \alpha\sqrt{1+2\delta}, \tag{66}$$

and the combined anisotropy coefficient

$$\eta = \frac{\epsilon - \delta}{1+2\delta}, \tag{67}$$

equation (65) can be rewritten as

$$(1+2\eta)V_{NMO}^2\|\widehat{\mathbf{p}}\|^2 + \left(1 - 2\eta V_{NMO}^2\|\widehat{\mathbf{p}}\|^2\right)\alpha^2 p_3^2 = 1. \tag{68}$$

This is exactly the equation derived by Alkhalifah (2000) using the dispersion relation for q-P waves.

Propagation of pseudo-S waves

Let us now study the possibility for the existence of S waves in a pseudo-acoustic VTI medium. We recognize immediately from equation (62) that $\Lambda_3 = 1$ cannot be satisfied, i.e., that no SH-wave propagation is possible. However, the situation is less clear for Λ_2. We see from equation (63) that for elliptical anisotropy, i.e., $\epsilon = \delta$, we have $b = 0$, which implies $\Lambda_2 = 0$ and then there are no S waves in pseudo-acoustic elliptically anisotropic media. For $\delta > \epsilon$, we have $b > 0$, which makes the square root in equation (63) larger than a and thus $\Lambda_2 < 0$. Again, there is no propagation of S waves in this case. However, for $0 > b \geq -a^2$, the condition $\Lambda_2 = 1$ can be satisfied. In accordance with the literature (Alkhalifah, 2000; Grechka et al., 2004), we refer to the resulting wave propagation as pseudo-S wave.

To analyse the propagation of the pseudo-S wave, note that the condition $\Lambda_2 = 1$ can only be satisfied for real a and b if $a \geq 2$. This implies a rather low propagation velocity of the pseudo-S wave. If the condition $a \geq 2$ is satisfied, the pseudo-S waves need to satisfy the very same eikonal equation (65) as the q-P waves. This conclusion coincides with the observation of Amazonas et al. (2010), who attribute the pseudo-S wave to the fact that the dispersion relation of equation (65) becomes real again for very large propagation angles or very small propagation velocities.

DISCUSSION

At this point, the advantages of our method are evident as compared to methods that derive the eikonal and wave equations in VTI media using dispersion relations (Alkhalifah, 2000; Pestana et al., 2011). It proves that the q-P wave and the pseudo-S wave are governed by the same eikonal equation (65) and that the latter can only exist for $\eta > 0$. Moreover, from our derivation, it has become clear that the eikonal equation (65) or its version (68) of Alkhalifah (2000) is valid for small δ but arbitrary ϵ and γ. Additionally, in the process, we have derived the ray-tracing system (54)–(56) that allows to trace rays in weakly anisotropic VTI media.

As an example, Figure 1 shows the q-P wavefront in the homogeneous pseudo-acoustic case obtained by tracing rays (red lines) using system (54)–(56) for two cases of weakly anisotropic VTI media. Finally, having at our disposal the general transport equation, we can determine the approximate amplitude behaviour not only of the q-P waves but also of the pseudo-S wave, simply using equation (59) with $\beta = 0$.

Pestana et al. (2011) derived independent q-P and q-SV wave equations using the dispersion relations of Tsvankin (1996). They came to the same conclusion as Fei and Liner (2008) that in VTI media where $\epsilon \approx \delta$ we can use the acoustic quasi-P wave without the influence of the quasi-SV wave. In our work, we show that in fact this condition is very natural and can be concluded directly from the eigenvalues of the Christoffel matrix. This derivation demonstrated that there is no q-SV wave in an elliptic medium with $\epsilon = \delta$ or in any VTI medium with $\eta \leq 0$.

CONCLUSIONS

In this work, we have derived a complete wave equation for weakly anisotropic VTI media by linearizing the stiffness matrix right from the start. For most parts of the derivations, only linearization in Thomsen parameter δ is sufficient, allowing allowing arbitrary values for ϵ and δ. The resulting wave equation is a

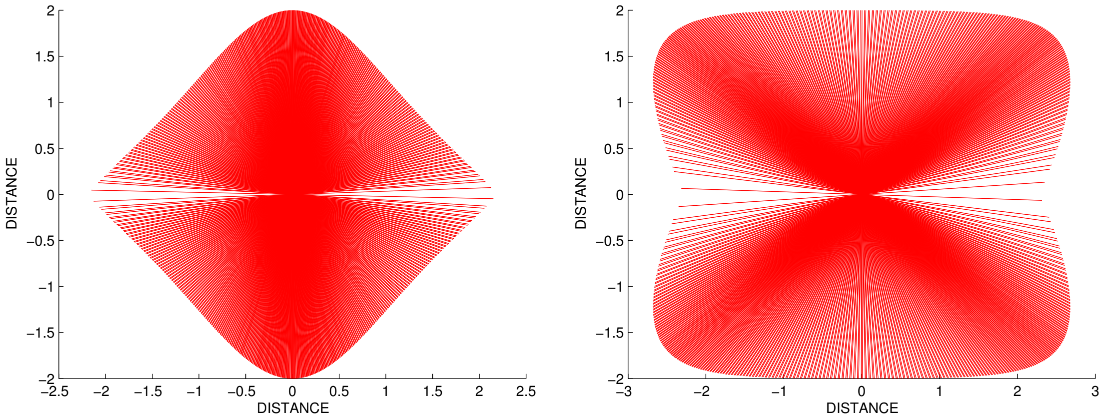

Figure 1: The q-P wavefront in the homogeneous pseudo-acoustic case obtained by ray tracing (red lines) using system (54)–(56). Left: $\delta = -0.2$ and $\epsilon = 0.10$. Right: $\delta = 0.5$ and $\epsilon = 0.15$.

system of three coupled differential equations for the three components of the dislocation vector. For $\delta = 0$, the third equation becomes an independent equation for the third component of the particle displacement, identical to the isotropic situation, and the first two equations remain coupled.

The solution of the wave equation with the linearized stiffness tensor using ray theory provides the corresponding eikonal and transport equations for q-P and q-S waves in weakly anisotropic VTI media. In the pseudo-acoustic case ($\beta = 0$), the eikonal equation for q-P waves reduces to the one of Alkhalifah (2000). The study of the q-SV waves in the pseudo-acoustic case demonstrated that these waves can exist only of $\eta > 0$ and that they must satisfy the same eikonal equation as the q-P waves.

ACKNOWLEDGEMENTS

This work was kindly supported by the National council of Scientific and Technological Development (CNPq), Brazil, and the sponsors of the Wave Inversion Technology (WIT) Consortium, Germany.

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APPENDIX A

WAVE EQUATION IN WEAKLY ANISOTROPIC VTI MEDIA

In this appendix we derive equation (20). We start from the following decomposition of the stiffness matrix of equation (4):

$$\mathbf{C}^{\mathrm{VTI}}_{\mathrm{weak}} = \mathbf{C}^{\mathrm{ISO}} + \mathbf{C}^{\epsilon} + \mathbf{C}^{\delta} + \mathbf{C}^{\gamma}. \tag{69}$$

where $\mathbf{C}^{\mathrm{ISO}}$ is given by equation (5) and the others are

$$\mathbf{C}^{\epsilon} = \begin{bmatrix} 2\epsilon(\lambda+2\mu) & 2\epsilon(\lambda+2\mu) & 0 & 0 & 0 & 0 \\ 2\epsilon(\lambda+2\mu) & 2\epsilon(\lambda+2\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{70}$$

$$\mathbf{C}^{\delta} = \begin{bmatrix} 0 & 0 & \delta(\lambda+2\mu) & 0 & 0 & 0 \\ 0 & 0 & \delta(\lambda+2\mu) & 0 & 0 & 0 \\ \delta(\lambda+2\mu) & \delta(\lambda+2\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{71}$$

$$\mathbf{C}^{\gamma} = \begin{bmatrix} 0 & -4\gamma\mu & 0 & 0 & 0 & 0 \\ -4\gamma\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu\gamma \end{bmatrix}. \tag{72}$$

By using the linearity of the derivative we can rewrite equation (1) as

$$\rho\frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial}{\partial x_j}\left(c^{\mathrm{ISO}}_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \frac{\partial}{\partial x_j}\left(c^{\delta}_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \frac{\partial}{\partial x_j}\left(c^{\epsilon}_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \frac{\partial}{\partial x_j}\left(c^{\gamma}_{ijkl}\frac{\partial u_k}{\partial x_l}\right). \quad (73)$$

For each of the terms, only a few components of the stiffness tensor are nonzero. For $\mathbf{C}^\epsilon$, the only nonzero components are

$$C^\epsilon_{11} = C^\epsilon_{22} = C^\epsilon_{12} = C^\epsilon_{21} = c^\epsilon_{1111} = c^\epsilon_{2222} = c^\epsilon_{1122} = c^\epsilon_{2211} = 2\epsilon(\lambda+2\mu)\ , \quad (74)$$

which leads to

$$\begin{aligned}\left[\frac{\partial}{\partial x_j}\left(c^{\epsilon}_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\right]_{i=1,2,3} &= \begin{bmatrix} 2\epsilon(\lambda+2\mu)\frac{\partial}{\partial x_1}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] + 2\frac{\partial\epsilon(\lambda+2\mu)}{\partial x_1}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] \\ 2\epsilon(\lambda+2\mu)\frac{\partial}{\partial x_2}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] + 2\frac{\partial\epsilon(\lambda+2\mu)}{\partial x_2}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] \\ 0 \end{bmatrix} \\ &= 2\widehat{\nabla}[\epsilon(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}], \end{aligned} \quad (75)$$

where $\widehat{\nabla}$ is given by equation (21). Correspondingly, the only nonzero components of the other matrices are

$$C^\delta_{13} = C^\delta_{31} = C^\delta_{23} = C^\delta_{32} = c^\delta_{1133} = c^\delta_{3311} = c^\delta_{2233} = c^\delta_{3322} = \delta(\lambda+2\mu) \quad (76)$$

and

$$\begin{aligned} C^\gamma_{12} &= C^\gamma_{21} = c^\gamma_{1122} = c^\gamma_{2211} = -4\gamma\mu \\ C^\gamma_{66} &= c^\gamma_{1212} = c^\gamma_{2121} = c^\gamma_{1221} = c_{2112} = 2\gamma\mu. \end{aligned} \quad (77)$$

Therefore,

$$\begin{aligned}\left[\frac{\partial}{\partial x_j}\left(c^{\delta}_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\right]_{i=1,2,3} &= \begin{bmatrix} \delta(\lambda+2\mu)\frac{\partial^2 u_3}{\partial x_3\partial x_1} + \frac{\partial\delta(\lambda+2\mu)}{\partial x_1}\frac{\partial u_3}{\partial x_3} \\ \delta(\lambda+2\mu)\frac{\partial^2 u_3}{\partial x_3\partial x_2} + \frac{\partial\delta(\lambda+2\mu)}{\partial x_2}\frac{\partial u_3}{\partial x_3} \\ \delta(\lambda+2\mu)\frac{\partial}{\partial x_3}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] + \frac{\partial\delta(\lambda+2\mu)}{\partial x_3}\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right] \end{bmatrix} \\ &= \widehat{\nabla}[\delta(\lambda+2\mu)\nabla\cdot\mathbf{u}] + \nabla[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}] - 2\widehat{\nabla}[\delta(\lambda+2\mu)\widehat{\nabla}\cdot\mathbf{u}], \end{aligned} \quad (78)$$

and

$$\begin{aligned}\left[\frac{\partial}{\partial x_j}\left(c^{\gamma}_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\right]_{i=1,2,3} &= \begin{bmatrix} 2\frac{\partial}{\partial x_2}\left(\mu\gamma\left[\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right]\right) + 4\left[\frac{\partial u_2}{\partial x_1}\frac{\partial\mu\gamma}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial\mu\gamma}{\partial x_1}\right] \\ -2\frac{\partial}{\partial x_1}\left(\mu\gamma\left[\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right]\right) + 4\left[\frac{\partial u_1}{\partial x_2}\frac{\partial\mu\gamma}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial\mu\gamma}{\partial x_2}\right] \\ 0 \end{bmatrix} \\ &= 2\widehat{\nabla}^\perp[\mu\gamma\widehat{\nabla}^\perp\cdot\mathbf{u}] + 4\mathbf{J}\nabla\mathbf{u}\widehat{\nabla}^\perp[\mu\gamma] \end{aligned} \quad (79)$$

where $\widehat{\nabla}^\perp$ and $\mathbf{J}$ are given by equations (22) and (23), respectively.

Collection of the results of equations (75), (78) and (79) and substitution in equation (73) results in equation (20).

APPENDIX B

TRANSPORT EQUATION

In this appendix we detail the procedure for obtaining the general transport equation in a weakly anisotropic heterogeneous VTI medium. This derivation follows the procedure for an isotropic medium as detailed by Pujol (2003).

Starting from equations (27) and (28), we have

$$c_{ijkl}p_j p_l U_k = \rho\Lambda U_i, \tag{80}$$

where Λ are the eigenvalues of the Christoffel matrix and $p_m = \partial T/\partial x_m$. Taking the derivative of equation (80) with respect to p_m we obtain

$$c_{ijkl}(\delta_{lm}p_j + \delta_{jm}p_l)U_k + c_{ijkl}p_j p_l \frac{\partial U_k}{\partial p_m} = \rho\frac{\partial \Lambda}{\partial p_m}U_i + \rho\Lambda\frac{\partial U_i}{\partial p_m}, \tag{81}$$

where δ_{ij} denotes the Kronecker symbol, not to be confused with the anisotropy parameter δ. Multiplying the second term on the left side of equation (81) by U_i and summing over i yields, upon the exchange $i \leftrightarrow k$ and $j \leftrightarrow l$,

$$c_{ijkl}p_j p_l \frac{\partial U_k}{\partial p_m}U_i = c_{klij}p_j p_l \frac{\partial U_i}{\partial p_m}U_k = \rho\Lambda\frac{\partial U_i}{\partial p_m}U_i, \tag{82}$$

where we have used equation (80) and the symmetry of $c_{ijkl} = c_{klij}$. In the same way, multiplying the first term on the left side of equation (81) with U_i and summing over i yields

$$\begin{aligned} c_{ijkl}(\delta_{lm}p_j + \delta_{jm}p_l)U_i U_k &= (c_{ijkm}p_j + c_{imkl}p_l)U_i U_k \\ &= (c_{klim}p_l + c_{imkl}p_l)U_i U_k = 2c_{imkl}p_l U_i U_k, \end{aligned} \tag{83}$$

where we have used the exchange $i \leftrightarrow k$, replacement $j \to l$, and symmetry $c_{klim} = c_{imkl}$. Together with results (82) and (83), multiplication of equation (81) with U_i plus subsequent summation over i yields

$$c_{ijkl}U_i U_k p_l = \frac{\rho}{2}\|\mathbf{U}\|^2\frac{\partial \Lambda}{\partial p_j}. \tag{84}$$

Therefore,

$$\frac{\partial}{\partial x_j}\Big(c_{ijkl}U_i U_k p_l\Big) = \frac{\partial}{\partial x_j}\Big(\frac{\rho}{2}\|\mathbf{U}\|^2\frac{\partial \Lambda}{\partial p_j}\Big) = \nabla\cdot\Big(\frac{\rho}{2}\|\mathbf{U}\|^2\frac{\partial \Lambda}{\partial \mathbf{p}}\Big). \tag{85}$$

For a weakly anisotropic VTI medium, the derivatives of the eigenvalues can be calculated in a common form for all three wave types from equation (48). They are given by

$$\frac{\partial \Lambda}{\partial p_m} = 2v^2 p_m + 4\xi v^2(\delta_{1m}p_1 + \delta_{2m}p_2) + 4\zeta\alpha^2 v^2 p_m[\delta_{m3}\|\widehat{\mathbf{p}}\| + (\delta_{m1}+\delta_{m2})p_3^2], \tag{86}$$

or, in vectorial form (see equation (54)),

$$\frac{\partial \Lambda}{\partial \mathbf{p}} = 2v^2\left[\mathbf{p} + 2\xi\widehat{\mathbf{p}} + 2\zeta\alpha^2 p_3\begin{pmatrix} p_1 p_3 \\ p_2 p_3 \\ \|\widehat{\mathbf{p}}\|^2 \end{pmatrix}\right]. \tag{87}$$

Finally, substituting the above result in equation (85) yields equation (59).