INSTABILITIES IN PSEUDO-ACOUSTIC WAVE EQUATION MODELLING OF TTI MEDIA

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ABSTRACT

There is a need for reverse-time migration from TTI media in the hydrocarbon industry. Therefore, attempts have been made to develop pseudo-acoustic wave equations for TTI media. All of these wave equations work well if the orientation of the media's symmetry axes are aligned throughout the computational domain. However, in the general case of spatially varying symmetry axes numerical instabilities occur. Some authors claimed to be able to avoid these instabilities. They achieved this only by applying extensive smoothing of the angles of symmetry axis orientation or by using certain staggered grids which inherently apply some averaging. However, when smoothing, the advantages of grid based forward modelling schemes, e.g. sharp velocity contrasts at interfaces or sharp changes of the symmetry axes, are lost. Modifying the elastodynamic equations for anisotropic media for the acoustic case leads to numerically stable results.

INTRODUCTION

For many years reverse-time migration (RTM) was based on the pure acoustic wave equation. In certain cases, however, the acoustic wave equation is not appropriate since to a certain amount seismic anisotropy may be present in seismic recordings. Therefore, there is an increasing demand that anisotropic reverse-time migration needs to be applied to such kind of data. Unfortunately, there is no anisotropic wave equation *per se* which could replace the ordinary (usually constant density) wave equation.

Alkhalifah (2000) presented an acoustic wave equation for anisotropic media based on his acoustic approximation for TI media (Alkhalifah, 1998) . This wave equation is a PDE of 4th order in time, which can be represented in the form of two coupled PDEs of 2nd order.

The reasons to deal with the acoustic approximation for TI media for RTM are that (1) the formulas are simpler and (2) less computer resources (i.e. memory and CPU time) are required than for the full elastic anisotropic equations of motion; (3) qS-wave energy propagation is significantly reduced, however, undesired artifacts are present in the numerical solutions. This reflects the fact that there is coupling between qP- and qS-waves in anisotropic media and an acoustic approximation does not exist.

After Alkhalifah several other authors, e.g. Zhou et al. (2006a), Hestholm (2007), Du et al. (2008), Duveneck et al. (2008), came up with variants of the so-called pseudo-acoustic wave equation for anisotropic media. These equations differ in the way in which auxiliary variables are introduced in order to split the equation into a system of two coupled PDEs. All of these wave equations have in common that they are initially formulated in the frequency-wavenumber domain assuming that the S-wave velocity vanishes. After this, the equations are transformed to the time-space domain.

However, the equations are not applicable to media with tilt (against vertical) tilted symmetry axes. Therefore, coordinate transforms were introduced into these wave equations. Pseudo-acoustic wave equations for tilted TI (TTI) media were described and applied by, e.g., Zhang et al. (2003), Du et al. (2005), Zhou et al. (2006b), Operto et al. (2007), Fletcher et al. (2008), Lesage et al. (2008b), Lesage et al. (2008a), Zhang and Zhang (2009), Liu et al. (2009). These equations work well if the tilt angle is constant throughout the entire computational grid. The pseudo-acoustic wave equations for TTI media, however, lead to unstable modelling algorithms in cases where the symmetry axes are not fixed within the entire computational domain. These instabilities are due to the fact, that though the methods are consistent within each region of constant symmetry axis tilt, it no longer is consistent when regions with different orientations of the symmetry axis are treated within one computational domain. A reason is that the physical interpretation of the quantities in the pseudo-acoustic wave equations are, e.g., vertical and horizontal stresses (e.g. Duveneck et al. (2008)), respectively. However, after individual rotation by different tilt angles this interpretation is not consistent any more because the reference systems of the different media differ.

This result was found when the above mentioned equations were numerically solved by a centered grid Fourier method, which is of high accuracy and where the stability limits are known. The centered grid Fourier method does not require any smoothing or averaging. It therefore allows sharp contrasts in seismic velocities and also of the kind of anisotropy. Also in Zhang and Zhang (2008) and Zhang and Zhang (2009) instabilities were mentioned.

Since instabilities are obviously unavoidable in case of varying orientations of symmetry axes we suggest to use the full anisotropic equations of motion, where certain elastic constants, which determine qS-wave propagation in TI media, are canceled. This is the same starting point as in Duveneck et al. (2008). Here, however, no further approximation is made.

We first describe how to obtain the acoustic wave field from the elastodynamic equations for anisotropic media and how it simplifies for TI models. Numerical results for these equations are presented and compared to modelling results of a pseudo-acoustic wave equation.

EQUATIONS OF MOTION

The full anisotropic equations of motion in Cartesian coordinates read:

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i.$$

Here ϱ is density per volume, u_i are the displacement components, σ_{ij} are the stress tensor elements and f_i are the components of the body forces density. Time is denoted by t and x_j are the coordinate directions.

The stress-strain relation for VTI media using the elasticity matrix (in Voigt notation) is given by:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix}, \qquad (1)$$

with $\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$ the strain tensor elements and C_{mn} the elasticity matrix components, where five elements are independent and $C_{66} = \frac{1}{2}(C_{11} - C_{12})$. The modified elasticity matrix according to Duveneck et al. (2008) reads:

$$\varrho v_p^2 \begin{pmatrix} 1+2\epsilon & 1+2\epsilon & \sqrt{1+2\delta} & 0 & 0 & 0 \\ & 1+2\epsilon & \sqrt{1+2\delta} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}. \qquad (2)$$

where ϵ and δ are Thomsen's anisotropy parameters (Thomsen, 1986). Note that the components C_{44} and C_{66} vanish and $C_{12} = C_{11}$.

After rotation of the elasticity matrix in order to yield a tilted symmetry axis all elements of the elasticity matrix will in general be different from zero. In 3D it therefore is necessary to deal with all 21 components

Material	#[1]	v_p [m/s]	ϱ [kg/m^3]	ϵ	δ	tilt angle
Green River shale (202.71 MPa)	41	3292	2075	0.195	-0.220	0^o
Mesaverde shale (1599)	20	3901	2640	0.137	-0.012	-45^o
Mesaverde calcacer. sandstone (6423.6)	12	5460	2690	0.000	-0.264	45^o

Table 1: Materials used for the modelling examples (from top to bottom in the second example). [1] # refers to the order in Table 1 of Thomsen (1986).

of the elasticity matrix although only five are independent. Only in very specialized cases a smaller number of components can be sufficient. In 2D at maximum six elastic matrix components are sufficient.

MODELLING EXAMPLES

The 2D modelling examples were calculated using the pseudo-spectral Fourier method with a 2nd order finite-differences time stepping scheme. The model is made up of 405 $\times$ 405 grid nodes with spacings of 20 m. The time increment is 1 ms. The maximum frequency of the Ricker wavelet of the pressure source is 50 Hz. The source position is at (4040 m, 3600 m). The material parameters are taken from Table 1 of Thomsen (1986). All of the materials show substantial anisotropic behavior. The elasticity tensor is modified according to Equation (2) for each material so that qS-wave propagation is almost suppressed. After this, to obtain a tilted symmetry axis, the elasticity tensor is rotated according to

$$c'_{ijkl} = a_{im}\, a_{jn}\, a_{ko}\, a_{lp}\, c_{mnop} \tag{3}$$

where c and c' are the elasticity tensors before and after rotation, respectively, and a is the rotation matrix.

The first example (model 1) shows wave front snapshots at 900 ms propagation time for a homogeneous transversely isotropic medium (Green River shale, see Table 1) with a vertically oriented symmetry axis.

Figure 1 shows the horizontal displacement component, the vertical displacement component, the divergence of the displacement field and the curl of the displacement field, respectively. Note the diamond shaped S-wave artifacts inside the qP-wave front. Such artifacts are also present in modelling results obtained by the various pseudo-acoustic wave equations. Due to their low propagation speed, i.e. small wave length, the S-wave artifacts are spatially aliased, which causes their noisy appearance.

The second example (model 2) shows a series of wave front snapshots at 225, 450, 675 and 900 ms propagation time for a horizontally layered medium. (Figure 2) . The horizontal layer boundaries are located at 2000 m and 5000 m depth. The model consists of three different homogeneous materials (see Table 1) with differently oriented symmetry axes (0, 45 and -45 degrees). Note that S-wave artifacts are generated where the wave front hits an interface. Here the divergence of the displacement field is displayed. This is the quantity which corresponds to the field variable in the ordinary acoustic wave equation, i.e. the pressure.

As an example of numerical failure of a pseudo-acoustic wave equation a result of modelling with the equation of Fletcher et al. (2008) is shown in Figure 3. The same material parameters as for model 2 were used. In this case, instabilities develop at a very early time of the numerical modelling. (The modelling results are stable only if the orientations of the symmetry axes were the same in all layers.)

CONCLUSIONS

It was shown by modelling examples that it is possible to calculate the qP-wave field in TTI media with arbitrarily oriented symmetry axes by the full anisotropic elastodynamic equations without numerical instabilities. This is achieved by modifying the elasticity matrix in such a way that S-wave propagation is reduced. This method appears superior compared to pseudo-acoustic wave equations since it is numerically stable even if there are rapid changes of the orientation of the symmetry axes. Pseudo-acoustic wave equations so far known fail in this respect. However, implementations with the anisotropic equations of motion are computationally more costly than those with pseudo-acoustic wave equations. In our 2D implementations the elastodynamic code is roughly 50% more expensive with respect to both, CPU time

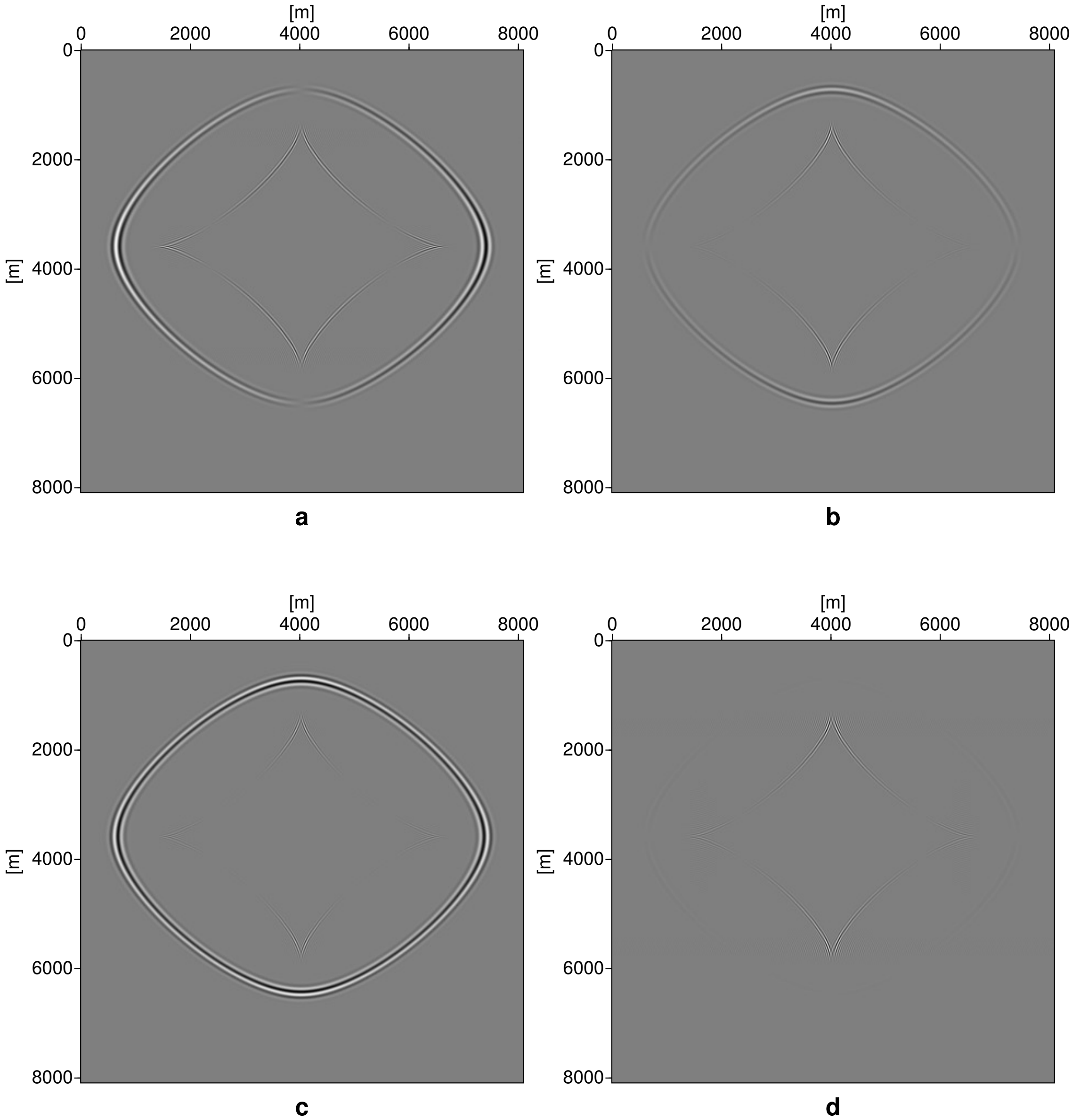

Figure 1: Wave field snapshots of a TI medium with vertical symmetry axis (model 1) at 900 ms propagation time: a: horizontal displacement component, b: vertical displacement component, c: divergence of the displacement field, d: curl of the displacement field. S-wave artifacts are visible.

and memory requirements, than the pseudo-acoustic code. We expect that this ratio is more severe in 3D implementations.

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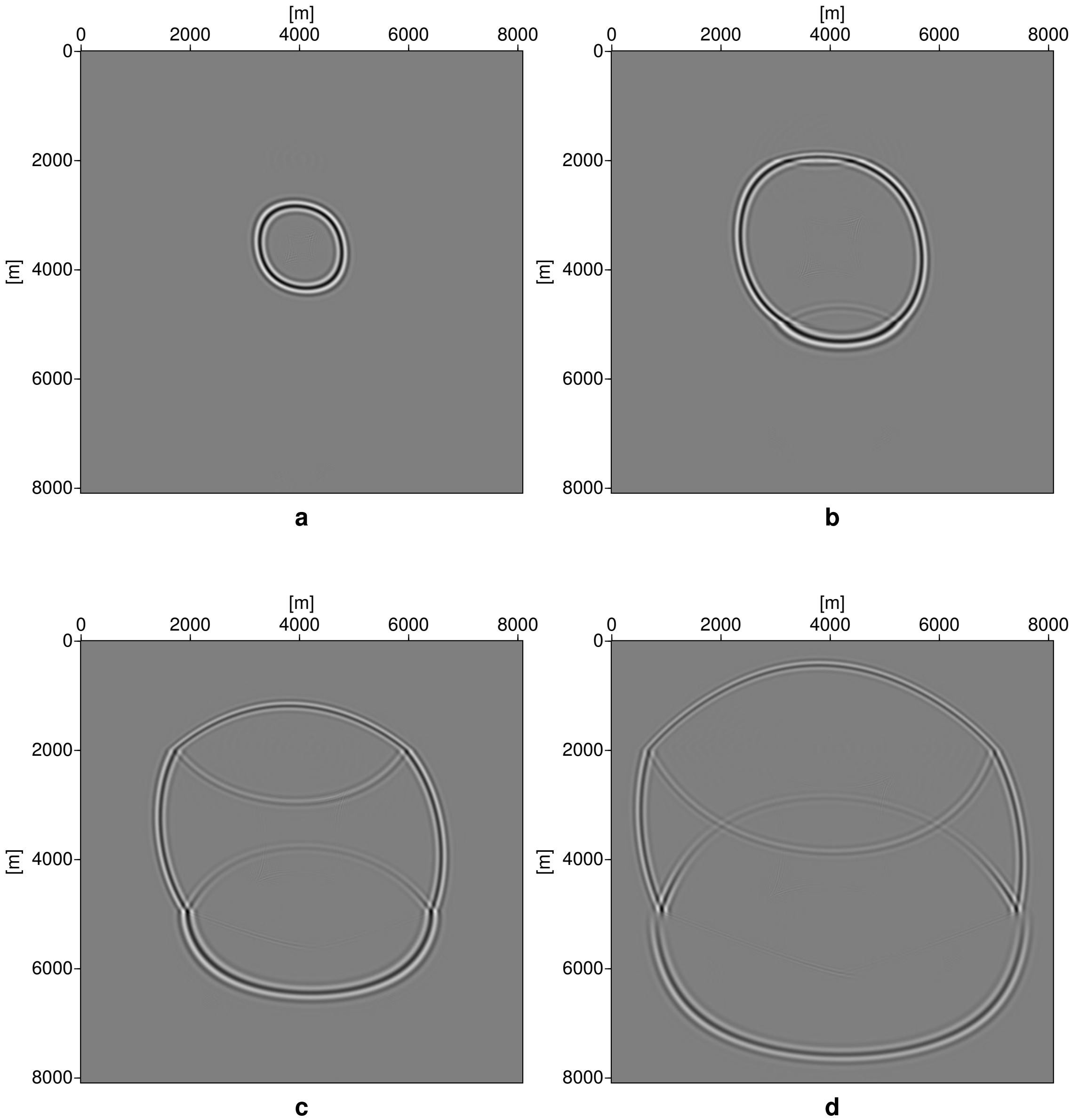

Figure 2: Snapshots of the divergence of the displacement field of a three-layer TI medium with varying orientations of the symmetry axes (model 2) at different propagation times: a: 225 ms, b: 450 ms, c: 675 ms, d: 900 ms. Results are numerically stable.

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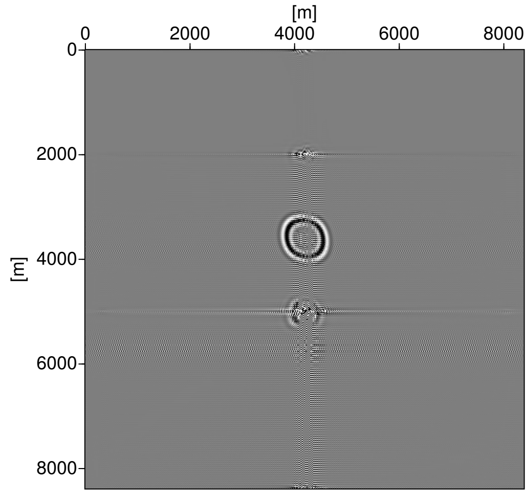

Figure 3: Snapshot from modelling results of model 2 using a pseudo-acoustic wave equation at 300 ms propagation time. Here, numerical instabilities are already observable.

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