APPLICATION OF SNELL'S LAW IN WEAKLY ANISOTROPIC MEDIA

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ABSTRACT

Evaluating Snell's law at an interface between two anisotropic media is a complicated problem that requires the numerical solution of a sixth-order polynomial. If the anisotropy is weak, an approximate solution can be obtained from iterative application of an expression derived with first-order perturbation theory. The result converges against the weak anisotropy solution and provides overall good accuracy.

INTRODUCTION

A wave impinging on a boundary between two elastic media will change its direction such that the horizontal slowness remains constant. This statement is, in essence, Snell's law. Its evaluation in isotropic media is simple and straightforward when the incidence angle and the velocities are known. In anisotropic media, however, where the velocities are directionally dependent, the determination of the vertical slowness involves the solution of a sixth-order polynomial. Furthermore, the six roots of the polynomial need to be assigned to the vertical slownesses of the three reflected and three transmitted waves. Since the direction of energy flow may differ considerably from the orientation of the slowness vector, this task is by no means trivial.

Snell's law has a large variety of applications in imaging and modelling of reflection seismic data. It comes in different guises, like the search for the vertical slowness, the determination of the reflection or transmission angle, or the phase velocity of a reflected or transmitted event. All of these are equivalent and laborious to solve as soon as anisotropy has to be taken into account.

Computations in anisotropic media are generally cumbersome, and isotropic methods usually fail to provide suitable solutions. Since, however, it has been observed that the anisotropy is often weak, a multifold of approximations have been suggested over the years in an attempt to simplify the computations. Many of these works focus on transversely isotropic media with a vertical symmetry axis (VTI media). A representative overview of approximate expressions for the phase velocities in VTI media is given, for example, by Fowler (2003).

Furthermore, in VTI media, Snell's law can be solved analytically, i.e. the exact value of the vertical slowness can be computed from the horizontal component (Červený and Pšenčík, 1972). In order to avoid the computationally-expensive square root in the resulting expression, several authors have introduced approximations for the slowness components, see, e.g., Schoenberg and de Hoop (2000) or Pedersen et al. (2007), and the references therein.

For media with arbitrary symmetry, the perturbation approach can be applied. Here, the medium is divided into a suitable background or reference part and a second part that contains the deviations or perturbations of the real medium from the background. Examples for reference media are isotropic media or anisotropic media with a higher symmetry, e.g. VTI. The computations are carried out in the background medium, and then correction terms using the perturbations are added.

Applications of first-order perturbation method were suggested by Červený and Jech (1982) and Jech and Pšenčík (1989). In the latter work, the authors already noted the potential of an iterative procedure to determine the vertical slowness, but have never applied their idea. It is also possible to consider higher order perturbation theory; this has been shown by, e.g., Farra (2001). Most authors focus on the determination of approximate expressions for phase velocities. In contrast to these works, our aim is to directly solve for the vertical slowness.

In this paper, we introduce an an extension and applications of the iterative method suggested by Jech and Pšenčík (1989). We derive an alternate formulation that contains only the P- or S-wave velocity in the background medium instead of the complete isotropic elastic tensor. This property simplifies the updating of the isotropic velocities required for the iteration process. Furthermore, we prove that the iteration converges against the weak anisotropy solution.

After a brief summary of the framework of first-order perturbation theory provided by Jech and Pšenčík (1989), we introduce our extension of that method. In the section thereafter, we apply the technique and show examples with different initial velocities to demonstrate the accuracy of the procedure.

METHOD

First-order perturbation method

In first-order perturbation method, the density-normalised elastic parameters of the anisotropic medium, a_{ijkl}, are represented by the sum of the elastic parameters of a suitable background medium, $a^{(0)}_{ijkl}$, and perturbations Δa_{ijkl}:

$$a_{ijkl} = a^{(0)}_{ijkl} + \Delta a_{ijkl} \quad . \tag{1}$$

Because the perturbations are assumed to be small, the first-order perturbation method yields an approximation for weak anisotropy. In this work, we will consider only isotropic background media, where the elastic tensor is given by

$$a^{(0)}_{ijkl} = (\alpha^2 - 2\,\beta^2)\,\delta_{ij}\delta_{kl} + \beta^2\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad . \tag{2}$$

In equation (2), α and β are the isotropic velocities of compressional and shear waves, respectively.

The aim of this work is the determination of the full slowness vector $\vec{p}$ in an anisotropic medium. In terms of the first-order perturbation method, $\vec{p}$ is represented by the sum of the slowness vector in the isotropic background, $\vec{p}^{(0)}$, and its perturbation $\Delta\vec{p}$.

Snell's law states that the horizontal slowness with respect to an interface is preserved. This requires that the perturbation appears only in the vertical component. Note that in this work, we use the terms 'horizontal' and 'vertical' with respect to the interface, represented by its normal vector $\vec{z}$. In this sense, the registration surface can also be considered to be an interface. The anisotropic slowness vector in the perturbed medium, $\vec{p}$, is thus written as

$$\vec{p} = \vec{p}^{(0)} + \vec{\Delta p} = \vec{p}^{(0)} + \Delta p\,\vec{z} \quad , \tag{3}$$

where the scalar quantity Δp is the perturbation of the vertical slowness.

As we apply a result by Jech and Pšenčík (1989) to relate the perturbation of the slowness to the perturbation of the medium parameters, we will now give a short summary of their derivation leading to the equations serving as basis for our work.

Let $\vec{g}^{(0m)}$ denote the polarisation vectors in the unperturbed medium (indicated by the superscript (0)). The index m defines the wavetype, where $m = 3$ is a P-wave and $m = 1, 2$ are S-waves. In the isotropic case, the polarisation vector of a P-wave equals the phase normal $\vec{n}$. The S-wave polarisation vectors are not unique since their only condition is that they are orthogonal to each other and lie in the plane perpendicular to $\vec{n}$. However, the degeneration of the S-wave polarisation can be removed with first-order perturbation theory. Consider the two vectors $\vec{e}^{(1)}$ and $\vec{e}^{(2)}$, where

$$\vec{e}^{(1)} = \begin{pmatrix} \cos\vartheta\, \cos\varphi \\ \cos\vartheta\, \sin\varphi \\ -\sin\varphi \end{pmatrix} \qquad \text{and} \qquad \vec{e}^{(2)} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \quad .$$

These vectors form an orthonormal base with $\vec{e}^{(3)} = \vec{g}^{(03)} = \vec{n}$. Then, the polarisation vectors of the S-waves in the unperturbed, i.e. background, medium are given by (Jech and Pšenčík, 1989)

$$\begin{aligned} \vec{g}^{(01)} &= \vec{e}^{(1)}\, \cos\chi + \vec{e}^{(2)}\, \sin\chi \\ \vec{g}^{(02)} &= -\,\vec{e}^{(1)}\, \sin\chi + \vec{e}^{(2)}\, \cos\chi \quad . \end{aligned} \tag{4}$$

The angle χ is determined from the weak anisotropy matrix (Pšenčík, 1998)

$$B_{MN} = \Delta a_{ijkl}\, e_i^{(M)}\, e_k^{(N)}\, n_j\, n_l \quad , \tag{5}$$

by

$$\tan 2\chi = \frac{2\, B_{12}}{B_{11} - B_{22}} \quad . \tag{6}$$

Jech and Pšenčík (1989) have shown that the first-order perturbation of the anisotropic eikonal equation,

$$G = 1 = a_{ijkl}\, p_i\, p_l\, g_j\, g_k \quad , \tag{7}$$

leads to

$$\Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} + 2\, a_{ijkl}^{(0)}\, \Delta p_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 0 \quad . \tag{8}$$

Using the representation of the perturbed slowness vector, equation (3), to express Δp_i and solving for the perturbation of the vertical slowness results in (Jech and Pšenčík, 1989)

$$\Delta p = -\frac{\Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}}{2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}} \quad . \tag{9}$$

Jech and Pšenčík (1989) have indicated that equation (9) can be used iteratively to increase the accuracy of the slowness vector but they do not pursue this approach any further. The procedure can be outlined as follows: assuming that suitable isotropic background velocities are available, the vertical slowness in the reference medium is computed from the isotropic eikonal equation by

$$p_3^{(0)} = \pm\sqrt{\frac{1}{V_0^2} - p_1^2 - p_2^2} \quad . \tag{10}$$

Here, the sign of p_3 is chosen according to whether we consider an incident or emerging wave, or a reflected or transmitted wave. After the first iteration step using equation (9), the background velocity for the next step is updated with

$$\frac{1}{V_0^2} = p_1^2 + p_2^2 + p_3^2 \quad . \tag{11}$$

Following this update, equation (9) is applied again, and the procedure is repeated until the desired accuracy is reached.

With the original expression for the slowness perturbation, equation (9), both P- and S-wave velocities have to be updated simultaneously even if only one wavetype is considered. In order to avoid this problem, we have developed an alternate formulation of equation (9), which contains only the isotropic velocity of the wavetype under consideration.

Separation of qP- and qS-waves

The equation for the perturbed slowness, (9), can be applied for qP- and qS-waves. In this section, we introduce a new formulation of equation (9) that depends only on α for the application to qP-waves, and only on β for the qS-waves. This formulation also includes an alternative expression for the weak anisotropy matrix $\underline{\mathbf{B}}$ defined in equation (5) to make the expression for shear wave polarisation required for the iteration given by equation (9) independent of α.

Substituting Δa_{ijkl} in the numerator of equation (9) we get

$$\begin{aligned}\Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} &= a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} - a_{ijkl}^{(0)}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} \\ &= a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} - 1 \quad , \end{aligned} \tag{12}$$

where the eikonal equation (7) was substituted for the isotropic background medium in the second term. For the denominator, we use the isotropic elasticity tensor, (2), and obtain

$$2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 2\,(\alpha^2 - \beta^2)\, z_i\, g_i^{(0)}\, p_l^{(0)}\, g_l^{(0)} + 2\,\beta^2\, z_i\, p_i^{(0)} \quad . \tag{13}$$

For P-waves in isotropic media, the polarisation vector is $\vec{g}^{(03)} = \vec{n}$. Furthermore, $\vec{p}^{(0)} = \vec{n}/\alpha = \vec{g}^{(03)}/\alpha$, and the expression for the denominator, (13), reduces to

$$2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 2\,\alpha^2\, z_i\, p_i^{(0)} \quad . \tag{14}$$

With this result for P-waves, equation (9) becomes

$$\Delta p = \frac{1 - a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}}{2\,\alpha^2\, z_i\, p_i^{(0)}} \quad . \tag{15}$$

To derive a corresponding expression for the S-waves, we make use of the fact that the polarisation vectors $\vec{g}^{(01)}$ and $\vec{g}^{(02)}$ are perpendicular to $\vec{n}$. We obtain

$$2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 2\,\beta^2\, z_i\, p_i^{(0)} \quad , \tag{16}$$

and thus,

$$\Delta p = \frac{1 - a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}}{2\,\beta^2\, z_i\, p_i^{(0)}} \quad . \tag{17}$$

In summary, equation (9) can be rewritten as

$$\Delta p = \frac{1 - a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}}{2\, V_0^2\, z_i\, p_i^{(0)}} \quad , \tag{18}$$

where V_0 is the isotropic background velocity of the wavetype under consideration, i.e. $V_0 = \alpha$ for qP-waves, and $V_0 = \beta$ for qS-waves.

With $\vec{g}^{(03)} = \vec{n}$ for P-waves, equation (18) is independent of β in this case. To make expression (18) independent of α for the application to qS-waves, we need to rewrite the weak anisotropy matrix $\underline{\mathbf{B}}$ such that it becomes also independent of α. This can be achieved by substituting the perturbation of the elastic tensor into the weak anisotropy matrix given by equation (5). With the isotropic elasticity tensor (2), we find that

$$\begin{aligned} a_{ijkl}^{(0)}\, e_i^{(M)}\, e_k^{(N)}\, n_j\, n_l &= (\alpha^2 - 2\,\beta^2)\, e_i^{(M)}\, n_i\, e_k^{(N)}\, n_k + \beta^2\,(e_i^{(M)}\, e_i^{(N)}\, n_k\, n_k + e_i^{(M)}\, n_i\, e_k^{(N)}\, n_k) \\ &= \beta^2\, \delta_{MN} \quad . \end{aligned} \tag{19}$$

Now B_{MN} can be expressed by

$$B_{MN} = a_{ijkl}\, e_i^{(M)}\, e_k^{(N)}\, n_j\, n_l - \beta^2\, \delta_{MN} \quad , \tag{20}$$

which is independent of α. With equations (18) and (20) we therefore have a new formulation of equation (9) that contains only the isotropic background velocity of the desired wavetype. This formulation considerably simplifies the updating of the reference velocities during the iteration since only one isotropic velocity needs to be updated if these new expressions are applied instead of equation (9).

However, as we will show in the next section, iterative application of equation (9) does not converge against the real anisotropic slowness vector, but against its weakly-anisotropic counterpart. Therefore, repeated application of equation (9) will only increase the accuracy within the weak anisotropy limit.

Convergence against the weak anisotropy approximation

Let us assume that the iterative determination of the perturbed slowness with equation (9) leads to the anisotropic slowness vector $\vec{p}$. In this case, after convergence has been achieved, we would expect Δp to be zero in further iteration steps, as $\vec{p}$ and $\vec{p}^{(0)}$ are then equal. Substituting the anisotropic eikonal equation (7) into expression (9), we find that

$$\Delta p = \frac{a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, (g_j\, g_k - g_j^{(0)}\, g_k^{(0)})}{2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}} \quad . \tag{21}$$

For a weakly anisotropic medium, the polarisation vectors $\vec{g}$ coincide with those in the isotropic background, $\vec{g}^{(0)}$, and thus, Δp becomes zero. Therefore, the iterative use of equation (9) converges against the slowness vector in the weak anisotropy approximation.

EXAMPLES

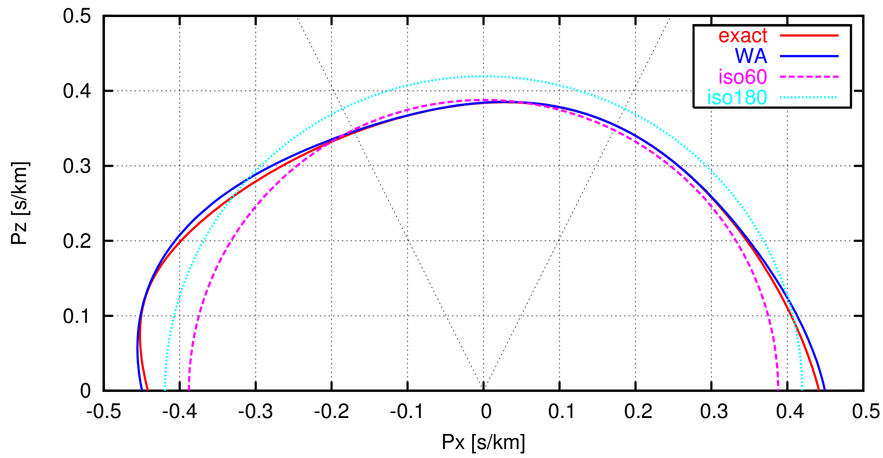
We have chosen an anisotropic medium with triclinic symmetry to demonstrate our method. The density-normalised elastic parameters of this rock, Vosges sandstone, (given in km^2/s^2) were taken from a paper by Mensch and Rasolofosaon (1997):

$$\underline{\mathbf{A}} = \begin{pmatrix} 4.95 & 0.43 & 0.62 & 0.67 & 0.52 & 0.38 \\ & 5.09 & 1.00 & 0.09 & -0.09 & -0.28 \\ & & 6.77 & 0.00 & -0.24 & -0.48 \\ & & & 2.45 & 0.00 & 0.09 \\ & & & & 2.88 & 0.00 \\ & & & & & 2.35 \end{pmatrix} \quad . \tag{22}$$

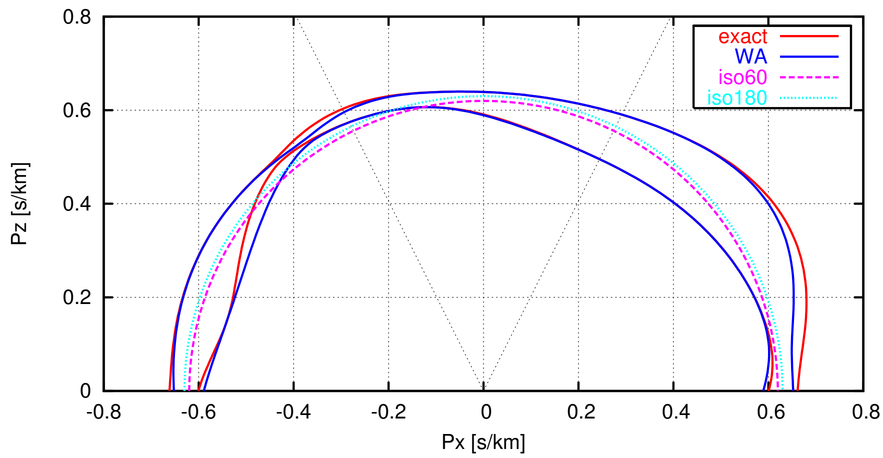
In our example, we have considered the azimuth angle $\phi = 0$. For this direction, the anisotropy of the qP-wave is about 20 % and it reaches up to 30 % for the shear waves. Although this rock does not qualify as a weakly anisotropic medium, it serves to investigate the range of applicability of our method.

The unit normal to the reference surface in our example, $\vec{z}$, coincides with the vertical direction (depth), i.e. $\vec{z} = (0, 0, 1)$. This means that we know the horizontal slowness components p_1 and p_2 and wish to determine the vertical slowness p_3 with the iteration procedure in equation (18).

The exact slowness surfaces as well as their approximation under the weak anisotropy assumption are shown in Figure (1). In order to find a suitable isotropic reference medium, we have computed sectorially best-fitting isotropic velocities. The determination procedure is an extension of the averaging method suggested by Fedorov (1968). It is briefly outlined in Appendix A. The slowness surfaces corresponding to two isotropic background velocities resulting from averaging over a range of 60° and 180° in inclination, respectively, are also shown in Figure (1).



(a) Slowness surfaces of the qP-wave



(b) Slowness surfaces of the qS-waves

Figure 1: Slowness surfaces for a rock with triclinic symmetry: the exact solution is given by the solid red lines; the solid blue lines show the approximation for weak anisotropy. The dashed pink and dotted light blue lines depict the sectorially best-fitting isotropic velocities averaged over an inclination range of 60° (indicated by the thin dotted straight lines) and 180°, respectively.

We have carried out the iteration procedure for qP- and qS-waves with two sets of initial background velocities. The first example uses the sectorially best-fitting velocities as initial reference medium that was obtained from an average over inclinations from -90 to 90 degrees with the fixed azimuth $\phi = 0$. The results of the first two iteration steps are shown in Figure (2); in Figure (2(a)) for the qP-wave and in Figure (2(b)) for the qS-waves. The weak anisotropy solution is also shown in these figures for comparison. We find that already after the second iteration, the numerical result matches the weak anisotropy solution rather well in most regions.

Closer inspection of the result shows that the vertical slowness is not well-behaved at larger inclination angles. There are two possible reasons for that behaviour. The first is that the denominator in equation (18) becomes zero for $p_3^{(0)} = 0$. We observe that small values of $p_3^{(0)}$ can lead to large correction terms Δp and thus make the algorithm unstable. Problems can also occur for larger horizontal slownesses when the background velocity is higher than the anisotropic phase velocity. It is even possible that $p_3^{(0)}$ becomes imaginary if the inverse of the background velocity is smaller than the horizontal slowness. For these phase directions, the initial velocity must be chosen smaller.

We have, therefore, repeated our experiment with initial background velocities that are just below the minimum phase velocities in the anisotropic medium to ensure that the vertical slowness in the initial isotropic reference medium is real and non-zero. The results from the first two iteration steps with this initial reference medium are displayed in Figures (3(a)) and (3(b)) for the qP- and qS-waves, respectively.

Our first observation in Figure (3) is that the convergence of vertical slowness is slower for these minimum velocities: the error after the second iteration is still larger than the error after the first iteration using the sectorially best-fitting velocities. Particularly for the faster $q\mathrm{S}_1$-wave the second iteration does not yield a satisfactory result. On the other hand, the problems at higher inclination angles have been partly resolved.

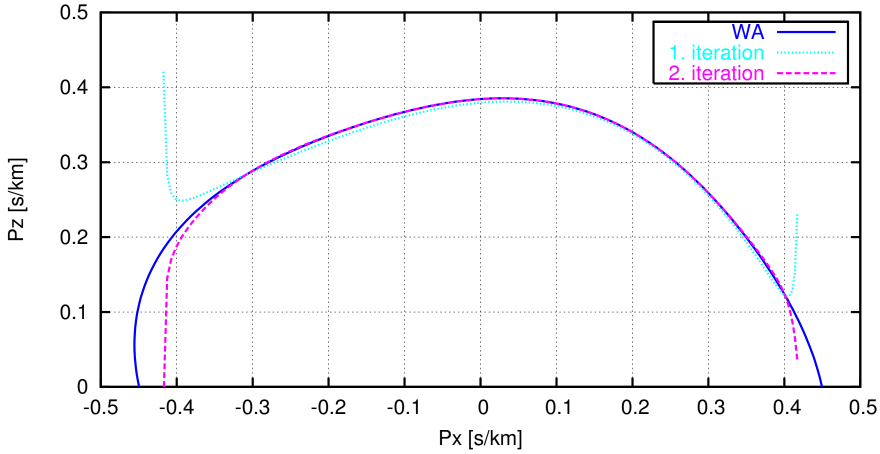
Since high incidence angles often lead to post-critical reflections, they would not be considered in reflection seismics. In this case, we suggest to use the sectorially best-fitting velocities as initial reference medium. Furthermore, when inclination angles are limited to a given range, the sectorial fit can be restricted to that range and might give a better approximation than a fit over 180°. For example, the 60° average for the qP-wave shown in Figure (1(a)) matches the true slowness surface much better than the 180° fit. In this case, already a single iteration would suffice. For the qS-waves, the situation is different as the isotropic velocity cannot properly approximate both anisotropic shear velocities at the same time. Therefore, a second iteration is recommended in any event. If high inclination angles need to be considered, minimum velocities should be chosen for the initial reference medium. In this case, however, a larger number of iteration steps is required.

CONCLUSIONS

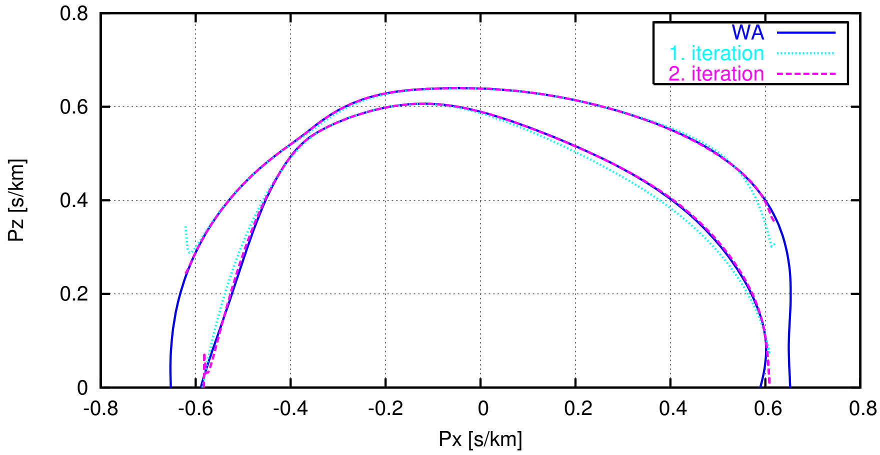
We have developed and applied an extension of a method originally suggested by Jech and Pšenčík (1989) to solve Snell's law in a weakly anisotropic medium. The objective of the method is the determination of the vertical slowness for a reflected or transmitted wave from the horizontal slowness components. The procedure is best implemented in an iterative fashion to increase the accuracy.

Since for many practical applications the anisotropy is weak, our method provides a valuable alternative to solving the sixth-order polynomial that would otherwise lead to the vertical slowness. Furthermore, the cumbersome assignation of the six roots to the three reflected and three transmitted events is no longer required because our algorithm allows the specification of one particular wavetype.

Depending on the choice of the initial background velocity, the iteration converges quickly; however, the resulting vertical slowness is that in the limit of the weak anisotropy approximation, and not the exact value. This may lead to significant deviations in regions with shear wave singularities. Due to numerical instabilities, the method is not reliable for high incidence angles.

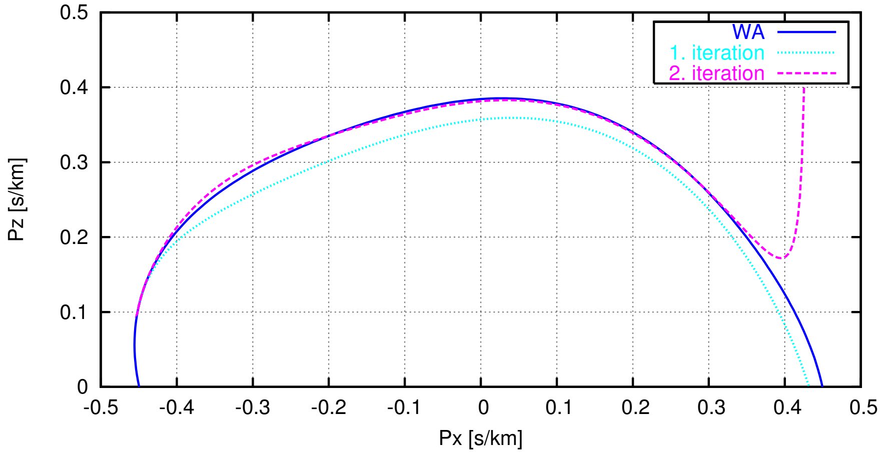


(a) Slowness surfaces of the qP-wave

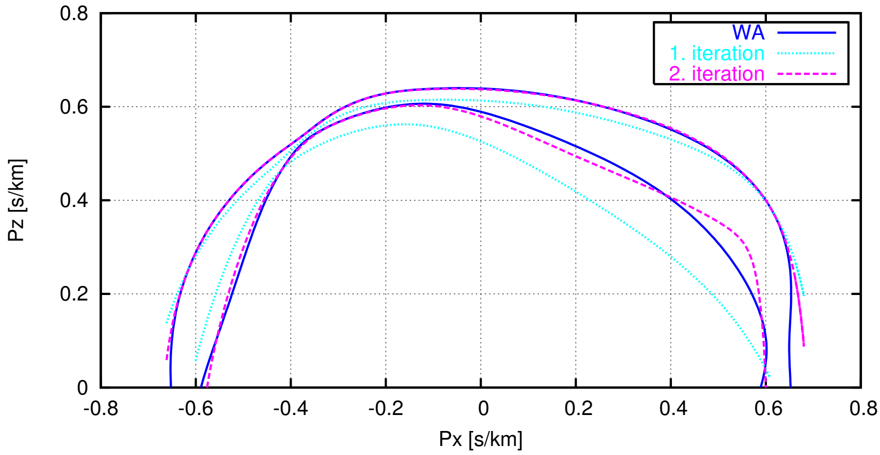


(b) Slowness surfaces of the qS-waves

Figure 2: Slowness surfaces for a rock with triclinic symmetry: the weak anisotropy solution is given by the solid blue lines; the dotted light blue and dashed pink lines depict the first and second iteration of our technique, respectively. In this example, the sectorially best-fitting isotropic background velocities were used as initial reference medium. The second iteration matches the weak anisotropy approximation closely, except for high inclination angles where the process becomes unstable (see text for details).



(a) Slowness surfaces of the qP-wave



(b) Slowness surfaces of the qS-waves

Figure 3: Slowness surfaces for a rock with triclinic symmetry: the weak anisotropy solution is given by the solid blue lines; the dotted light blue and dashed pink lines depict the first and second iteration of our technique, respectively. In this example, minimum velocities were used as initial reference medium. The stability problem has been reduced to some extent, but the convergence is slower than for the sectorially best-fitting velocities as initial background medium.

Outside these regions, a good coincidence between the analytic and numeric solution could be observed for the considered rock. We suggest to apply the method for imaging rather than modelling. In that case, the errors observed at large incidence angles are less meaningful since wide angle reflections are rarely taken into account.

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APPENDIX A

Determination of sectorially best-fitting isotropic background velocities

This appendix provides expressions for isotropic background media that yield the best fit to the anisotropic medium for a specified range or sector of phase directions. The derivation follows that for the globally best-fitting isotropic medium given in Fedorov (1968).

The phase velocities of the three wavetypes in an anisotropic medium are the eigenvalues of the Christoffel matrix $\underline{\mathbf{\Lambda}}$, with

$$\Lambda_{ik} = a_{ijkl}\, n_j\, n_l \quad , \tag{23}$$

where the a_{ijkl} are the elements of the density-normalised elasticity tensor and n_i is the phase normal.

According to Fedorov (1968), the isotropic medium with the corresponding Christoffel matrix $\underline{\mathbf{\Lambda}}^{(0)}$, for which

$$\langle (\Lambda_{ik} - \Lambda_{ik}^{(0)})^2 \rangle \stackrel{!}{=} \text{Min.} \tag{24}$$

becomes minimal is the best isotropic approximation of the anisotropic medium. The pointed brackets in equation (24) denote an averaging process over the phase directions that is described further below.

Minimisation of the objective function (24) leads to a linear system of equations with the solution

$$\begin{aligned} \alpha^2 &= a_{ijkl}\, \langle n_i\, n_j\, n_k\, n_l \rangle \quad , \\ \beta^2 &= \frac{1}{2}\left(a_{ijik}\, \langle n_j\, n_k \rangle - \alpha^2 \right) \quad . \end{aligned} \tag{25}$$

The resulting expressions for the isotropic P-wave velocity α and S-wave velocity β are averages over a given range of phase directions. Fedorov (1968) applied the following averaging operation for a function $A(\theta, \phi)$ over all phase directions,

$$\langle A \rangle = \frac{1}{4\,\pi} \int\limits_{\phi=0}^{2\,\pi} \int\limits_{\theta=0}^{\pi} A(\theta, \phi)\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \quad , \tag{26}$$

to obtain his well-known result for the globally best-fitting isotropic medium,

$$\begin{aligned} \alpha^2 &= \frac{1}{15}\,(a_{iikk} + 2\,a_{ikik}) \quad , \\ \beta^2 &= \frac{1}{30}\,(3\,a_{ikik} - a_{iikk}) \quad . \end{aligned} \tag{27}$$

Fedorov (1968) also determined the best-fitting isotropic velocities for the case that the direction of the phase normal is fixed, i.e. no averaging is carried out in equation (25). In this case, the resulting velocities coincide with those obtained by Backus (1965) from the weak anisotropy approximation. More precisely, the best-fitting isotropic shear wave velocity β is the quadratic mean (RMS) of the phase velocities of the qS$_1$- and qS$_2$-waves in the approximation for weak anisotropy, and (Backus, 1965; Fedorov, 1968)

$$\begin{aligned} \alpha^2 &= a_{ijkl}\, n_i\, n_j\, n_k\, n_l \quad , \\ \beta^2 &= \frac{1}{2}\left(a_{ijik}\, n_j\, n_k - \alpha^2\right) \quad . \end{aligned} \tag{28}$$

These results can be generalised to obtain an isotropic background medium that yields the best fit over a sector of phase directions. The sector is bounded by the inclination angles θ_1 and θ_2 and the azimuth angles ϕ_1 and ϕ_2. For this situation, the averaging operator

$$\langle A \rangle_{\theta,\phi} = \frac{\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} A(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi} \tag{29}$$

is applied to the expression for the velocities, (25), leading to the velocities for the sectorially best-fitting isotropic background medium. The explicit results for α and β are tedious due to the trigonometric expressions involved, but not difficult. One special case, the sectorially best-fitting P-wave velocity for a cone with the opening angle θ around the z-axis, i.e. $\phi_1 = 0$, $\phi_2 = 2\,\pi$, $\theta_1 = 0$, and $\theta_2 = \theta$ was also suggested by Ettrich et al. (2001).

Furthermore, it is possible, to average only over the inclination angle, for example, when the horizontal slowness, and therefore the azimuth angle ϕ is known. In that case the averaging operator reduces to

$$\langle A \rangle_{\theta} = \frac{\int_{\theta_1}^{\theta_2} A(\theta,\phi)\sin\theta\,\mathrm{d}\theta}{\int_{\theta_1}^{\theta_2} \sin\theta\,\mathrm{d}\theta} \quad . \tag{30}$$

Examples for sectorially best-fitting isotropic background velocities are shown in Vanelle and Gajewski (2004).

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