TRANSVERSE RAYLEIGH WAVE COMPONENTS

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ABSTRACT

The polarization of the Rayleigh wave in the simple case of an isotropic halfspace is known to be in the plane of propagation. Here we demontrate, based on previous work, that the solution of the equations of motion for this problem also contains transverse components. This is theoretically derived for the case of a horizontal single point force at the surface. We confirm this prediction using numerical modelling results for this case and make statements here: The amplitudes of these extra components are about 100 times smaller compared to the ordinary Rayleigh waves. Moreover, their amplitude decay with distance is faster.

INTRODUCTION

The Rayleigh wave propagates along the surface of an isotropic elastic halfspace without any dispersion. Its amplitude decays exponentially with depth. In the classical solution the Rayleigh wave is polarized in the plane of propagation and does not exhibit a transverse component.

However, in Kiselev (2004) a formal solution of the equation of motion was constructed which in addition to the classical case of Rayleigh wave polarization, e.g. Aki and Richards (1980), contains transverse components outside of the propagation plane. These transverse components have a lower frequency content compared to the main components of the Rayleigh wave. In the frame of the ray theory these components can be considered as additional components of the displacement of first order of the ray approximation. This paper, however, does not deal with ray theory.

In this paper the full theory of the Rayleigh wave is given for the example of the exact solution of the wavefield for the isotropic halfspace which explains these additional components, where the source is a horizontal single point force applied at the free surface. The derivation provides physical insight to the formal solution of Kiselev (2004). The wavefield of the Rayleigh wave is derived by calculation of residues of the exact solution. The analytical solution was tested by a Chebyshev forward modelling method (see, e.g. Kosloff et al. (1990), Tessmer (1995)). The additional transverse components of the Rayleigh wave are observed for certain directions in the synthetic modelling results.

In Appendix A the classical solution of Rayleigh wave propagation is constructed. Based on this the short derivation of the result from Kiselev (2004) is repeated in Appendix B. In Appendix C the derivation of the exact solution for a horizontal force in an elastic halfspace is presented.

A HORIZONTAL FORCE APPLIED TO AN ISOTROPIC HOMOGENEOUS HALF-SPACE

The source is introduced by the following condition at the free surface (x_1, x_2-plane):

$$\vec{T}(x_1, x_2) = \delta(x_1, x_2)\vec{i}_1 e^{i\omega t}.$$

Here $\vec{T}$ is the stress vector due to the source and $\vec{i}_1$ is the unit vector in x-direction. These boundary conditions define the unit horizontal force in x-direction. In Appendix C it is shown that the wavefield from this kind of a source is obtained by the following equations in cylindrical coordinates $(r, \varphi, x_3 \equiv z)$:

$$\begin{aligned} u_r = & \frac{\rho\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}H_0^{(2)}(\omega\zeta r)d\zeta \\ & -\frac{\rho\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta \\ & -\frac{\cos\varphi}{2\pi}\int_{0}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta, \end{aligned} \tag{1}$$

$$\begin{aligned} u_\varphi = & -\frac{\rho\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)]d\zeta \\ & +\frac{\omega\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}H_0^{(2)}(\omega\zeta r)d\zeta \\ & -\frac{\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta, \end{aligned} \tag{2}$$

$$\begin{aligned} u_3 = & -\frac{\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\zeta^2\frac{2\mu\alpha_p\alpha_s}{R}e^{-\omega\alpha_p x_3}H_1^{(2)}(\omega\zeta r)d\zeta \\ & +\frac{\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\zeta^2\frac{2\mu\zeta^2-\rho}{R}e^{-\omega\alpha_s x_3}H_1^{(2)}(\omega\zeta r)d\zeta. \end{aligned} \tag{3}$$

In the solution (1)-(3) u_r, u_φ, and u_3 are the components of displacement $\vec{u}$ in cylindrical coordinates, ω is the angular frequency, $H_i^{(k)}$ are Hankel functions of order i and kind k. ζ is the horizontal slowness and $\alpha_{p,s} = \sqrt{\zeta_2 - \frac{1}{v_{p,s}^2}}$ is the vertical slowness, where $v_{p,s}$ is either P-wave velocity v_p or S-wave velocity v_s, respectively.

The Rayleigh equation is given by:

$$R(\zeta) = \zeta^2\alpha_p\alpha_s - (\frac{1}{v_s^2} - 2\zeta^2)^2. \tag{4}$$

$\rho, v_p, v_s, \mu = \rho v_s^2$ are the elastic parameters of the halfspace.
The Rayleigh wave is described as the residue of the exact solution at the Rayleigh pole, i.e., the root of the Rayleigh equation. The components of the Rayleigh wave in cylindrical coordinates are given by

$$\begin{aligned} u_r^{(R)} = & \frac{\rho\omega\cos\varphi}{i}\Big[\frac{\zeta\alpha_s}{R'}e^{-\omega\alpha_p x_3}\Big](\zeta_R)H_0^{(2)}(\omega\zeta_R r) \\ & -\frac{\rho\cos\varphi}{ir}\Big[\frac{\alpha_s}{R'}e^{-\omega\alpha_p x_3}\Big](\zeta_R)H_1^{(2)}(\omega\zeta_R r), \end{aligned} \tag{5}$$

$$u_\varphi^{(R)} = -\frac{\rho\sin\varphi}{ir}\Big[\frac{\alpha_s}{R'}e^{-\omega\alpha_p x_3}\Big](\zeta_R)H_1^{(2)}(\omega\zeta_R r)], \tag{6}$$

$$\begin{aligned} u_3^{(R)} = & -\frac{\omega 2\mu\cos\varphi}{i}\Big[\zeta^2\frac{\alpha_p\alpha_s}{R'}e^{-\omega\alpha_p x_3}\Big](\zeta_R)H_1^{(2)}(\omega\zeta_R r) \\ & +\frac{\omega\cos\varphi}{i}\Big[\zeta^2\frac{2\mu\zeta^2-\rho}{R'}e^{-\omega\alpha_s x_3}\Big](\zeta_R)H_1^{(2)}(\omega\zeta_R r). \end{aligned} \tag{7}$$

In Eqs. (5)-(7) R' is the derivative of the Rayleigh equation, and ζ_R is the horizontal slowness of the Rayleigh wave.

From the integral representation and in Eq. (1) for the radial component u_r and Eq. (2) for the azimuthal component u_φ we find an interesting phenomenon: Setting $\phi = 0$, which means that we consider the direction in which the force is directed, we can see from Eq. 1 (third term) that there is a contribution of the S-wave in the wavefield. Therefore we have a wave which propagates with S-wave velocity. It is polarized like a P-wave, i.e. in radial direction.

We obtain a similar phenomenon for $\varphi = \frac{\pi}{2}$. This is the direction orthogonal to the force direction. We see from Eq. 2 (first term) that there is a contribution by the P-wave. We see a wave which propagates with P-wave velocity, which is polarized parallel to the wave front. In both cases these 'anormal' waves show r^{-1} decay with distance. Their frequency content is lower compared to the usual waves. This can be seen from the missing factor ω in front of the respective integrals.

In the next section we will confirm these conclusions derived from the analytical solution by numerical modelling using a pseudo-spectral Chebyshev method.

NUMERICAL MODELLING

Numerical modelling was performed using the pseudo-spectral Chebyshev method (Kosloff et al. (1990), Tessmer (1995)). The method delivers a solution to the equations of dynamic elasticity in a forward modelling manner. Spatial derivatives of the partial differential equations are calculated in the wave number domain using FFTs in the horizontal directions. The derivative with respect to the vertical direction are calculated in the Chebyshev transform space, also using FFTs. The time integration is based on a forth order Taylor expansion. The subsurface structure consists of a homogeneous elastic halfspace. The 3-D model is discretized on a numerical grid of $315 \times 315 \times 181$ grid points with a grid spacing of 10 m. The P- and S-wave velocities are 2000 m/s and 1155 m/s, respectively. The horizontal single point force has a Ricker-like time history with 50 Hz cut-off frequency (dominant frequency is 25 Hz). It is directed in the (horizontal) x-direction. The time step size of the modelling is 1.5 ms. The total propagation time is 1.8 s.

The source is located at grid location (105,105,3). It is located away from the lateral model boundaries to avoid contamination of the primary wavefield with boundary reflections. The source was placed slightly below the free surface, i.e. 3.6 m, to avoid numerical artifacts.

The wavefield was recorded at any grid node at the free surface for all time steps. The seismograms shown consist of ten traces covering an azimuth of 90^o at 1750 m lateral distance from the source. Since the modelling was done using a rectangular grid it was necessary to interpolate wavefield values between grid nodes to maintain the exact source-receiver distance. Interpolation was done by a 2D Fourier transform applying its shift property. Using the Fourier transform for interpolation is the most natural way, since the modelling algorithm is based on trigonometric expansion of the wavefield.

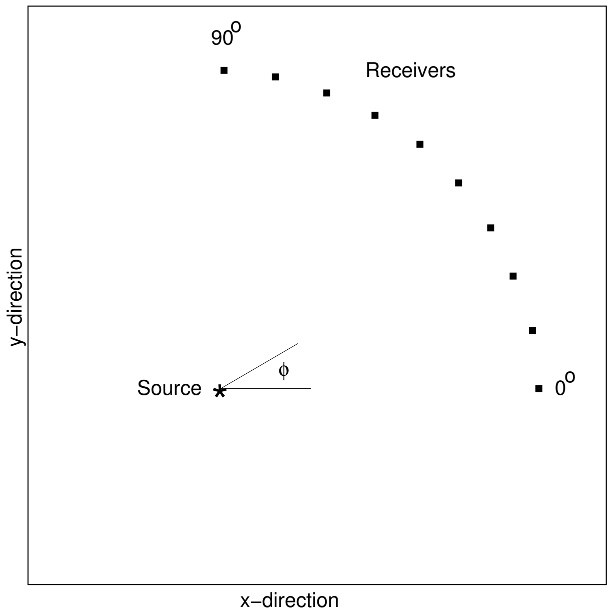

Figure 1: Geometry of source and receivers at various azimuths ϕ with a constant observation distance of 1750 m. The receiver azimuth increment is 10^o.

Basically, the wavefield is represented by three Cartesian components of the particle velocity (v_x, v_y, v_z).

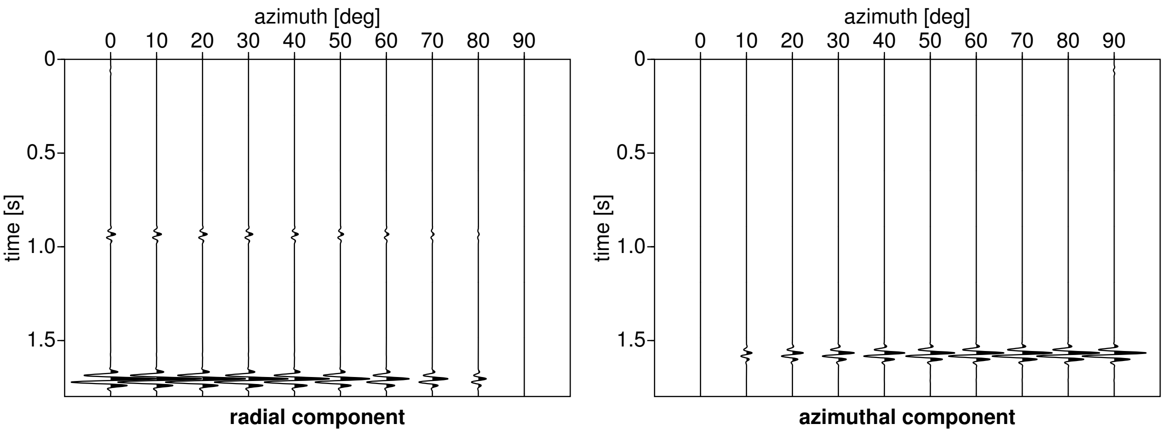

Figure 2: Seismograms recorded at 1750 m distance from the source with 10^o azimuth increments. Left: radial component; right: tangential component

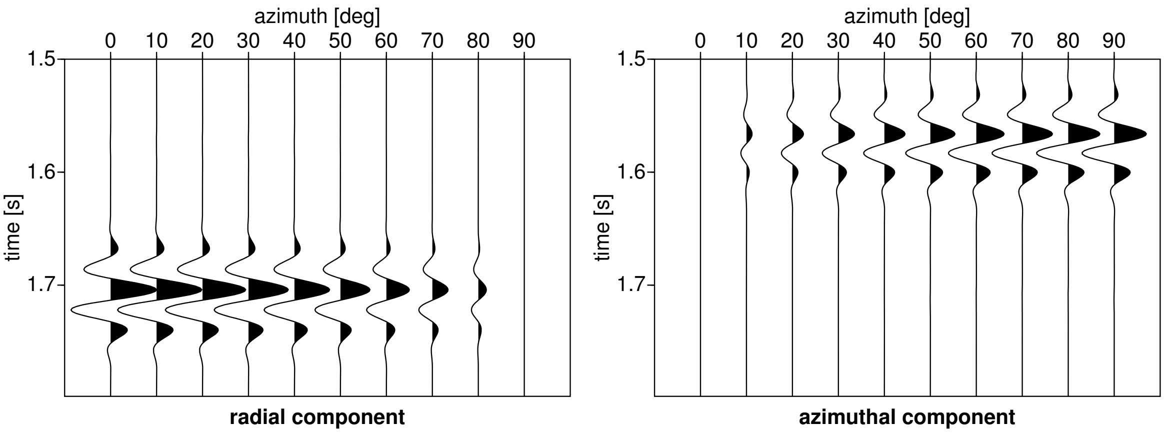

Figure 3: As Fig. 2, but time-windowed for S- and Rayleigh wave events.

In the seismogram displays we show only the horizontal components. They were rotated into the radial and tangential components. Ten seismograms with a 10^o azimuth increment are displayed in each section, where the azimuths are measured against the x-axis.

In Fig. 1 we show the geometrical setup of source and receiver positions. Fig. 2 shows the entire seismograms of the radial and tangential components of the surface recordings, where the amplification of the traces is scaled to the maximum value. In the radial component display the first and the later arrival correspond to the P- and Rayleigh wave, respectively. In the azimuthal component display only the S-wave event is visible. Fig. 3 shows the time window of the seismograms of Fig. 2, where the S-wave (earlier arrival) and the Rayleigh wave (later arrival) appear. Fig. 4 shows the same as Fig. 3, but with $10\times$ amplification. In the radial component there is a low-amplitude precursor (S-wave) before the Rayleigh wave event. In the azimuthal component a low-amplitude Rayleigh wave event is visible after the S-wave event.

Further amplified seismograms are shown in Fig. 5. Here the amplification compared to Fig. 3 is 100. In the tangential component seismogram at 90^o azimuth, the contribution in the tangential component of the additional term of the Rayleigh wave can be observed. It decays with decreasing azimuths. On the

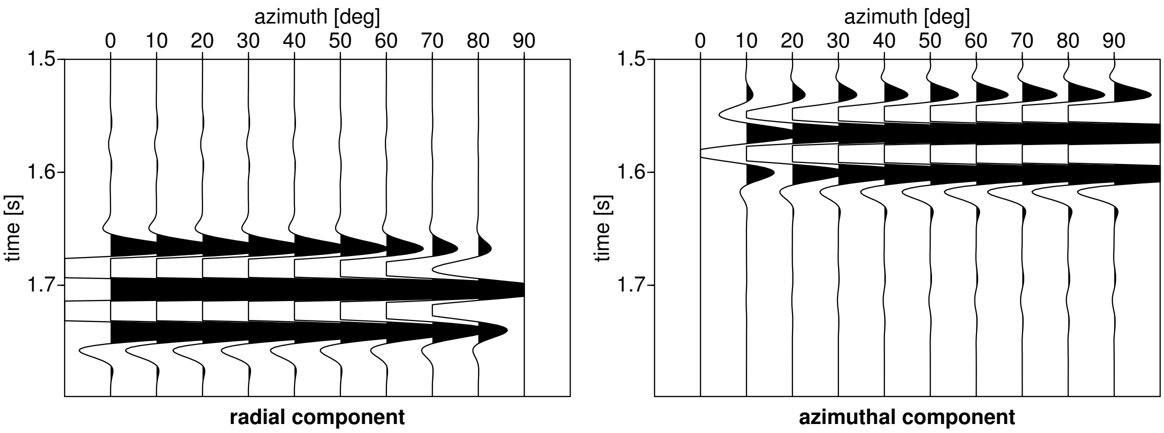

Figure 4: As Fig. 3, but 10x amplified.

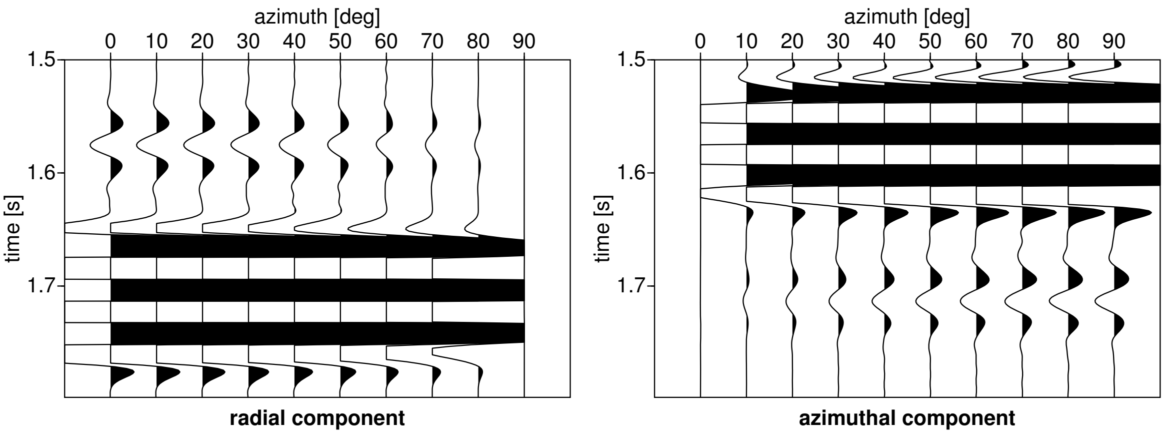

Figure 5: As Fig. 3, but 100x amplified.

other hand, in the radial component seismogram at 0^o azimuth, the contribution of the additional term of the S-wave can be observed. It decays with increasing azimuths.

The above described features of the wavefield can also be observed in the xy-plane snapshots (top view) at the surface. In Fig. 6 a snapshot of the x-component of the particle velocity field is shown at time 1.7625 s. The P-wave has already left the modelling area. Note the small amplitudes of the S-wave (outer wave front) in the x-direction and of the Rayleigh wave (strong next inner wave front) in the y-direction. The inner wave front is an undesired artifact which is a P-wave reflection form the bottom of the numerical grid. Other artifacts are due to imperfectly functioning absorbing boundary conditions.

A similar phenomenon as for S- and Rayleigh waves can also be observed on the P-wave front. In Fig. 7 a snapshot of the x-component of the particle velocity field is shown at time 0.4874 s. Note the small (but not vanishing) amplitudes of the P-wave (outer wave front) in the x-direction. The amplitudes of the additional terms of the P-wave are by an order of magnitude smaller than those of the S- and Rayleigh waves in this case.

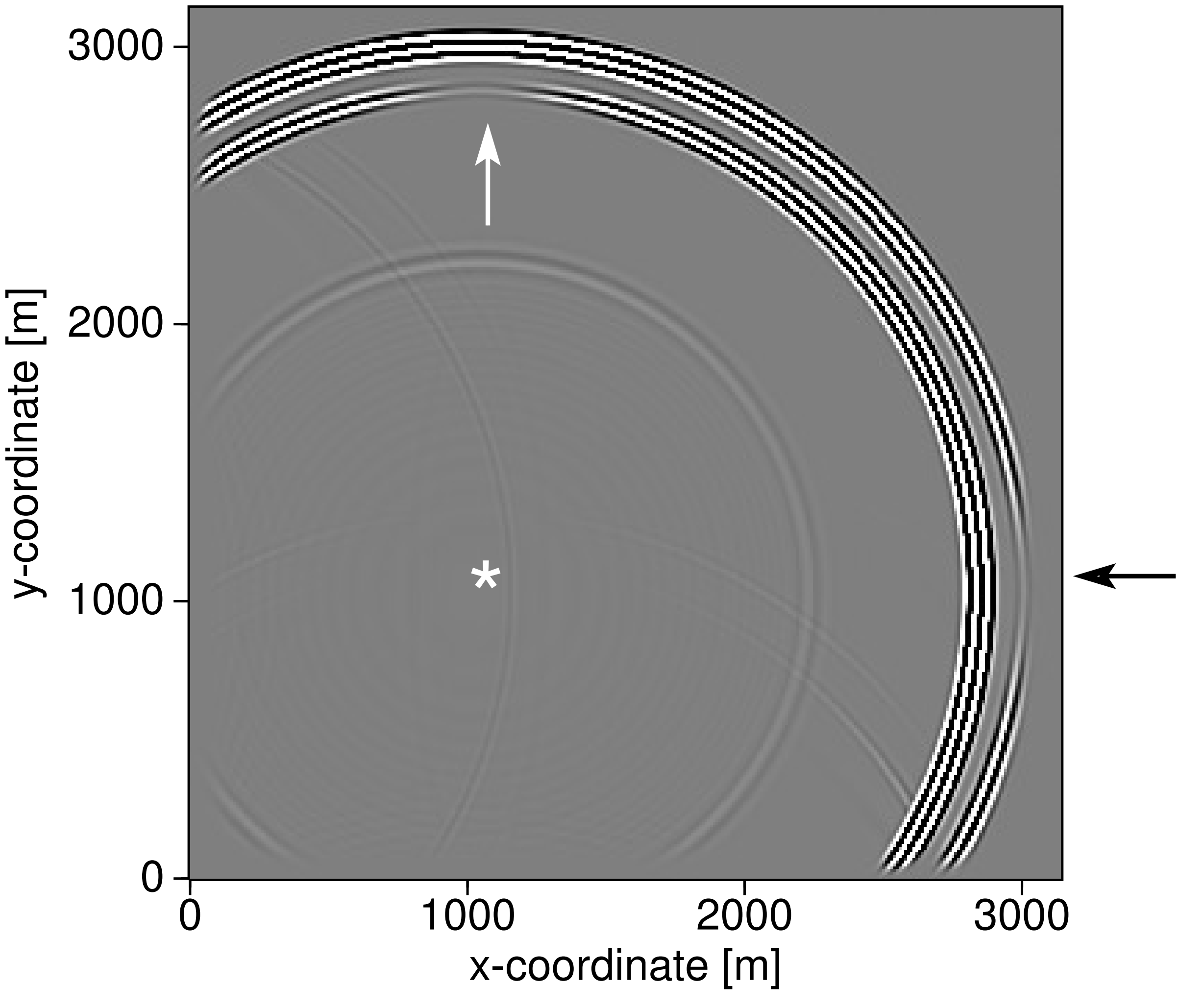

Figure 6: Snapshot of the x-component of the particle velocity at time=1.7625 s. Here small (but not vanishing) amplitudes of the S-wave in the x-direction (black arrow) and of the Rayleigh wave in the y-direction (white arrow) can be observed. The source location is indicated by a white asterisk. The other weak wavefronts are undesired artifacts.

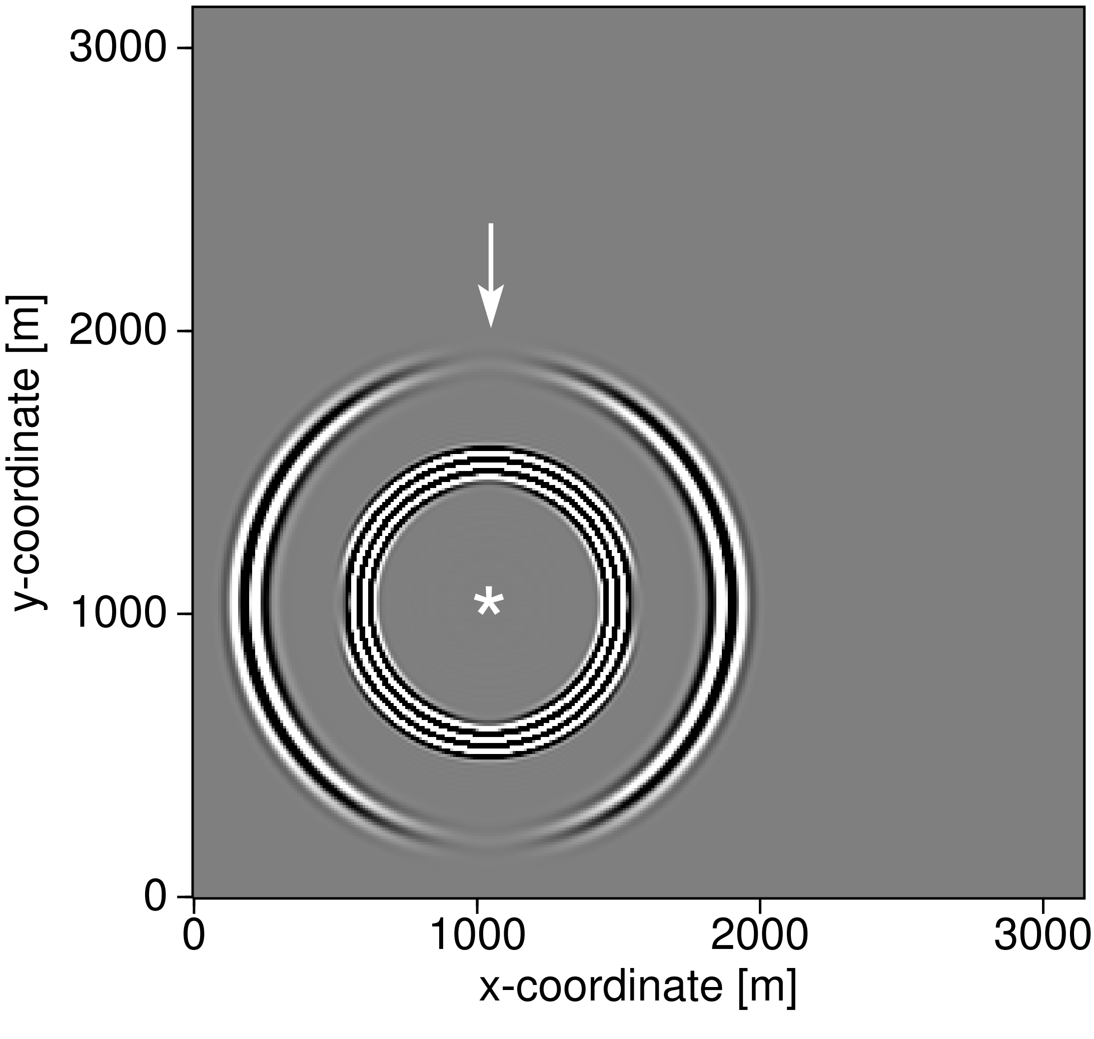

Figure 7: Snapshot of the x-component of the particle velocity at time=0.4875 s. Here small (but not vanishing) amplitudes of the P-wave in the y-direction can be observed (white arrow).

CONCLUSIONS

We examined the case of wave propagation in an isotropic elastic halfspace due to a horizontal single point force. The radial and tangential components of the Rayleigh wave have a main term and additional terms. In a certain direction the Rayleigh wave comprises only the additional term. Therefore, the Rayleigh wave is transversely polarized in this situation. In general the additional terms have considerably lower amplitudes than the main terms (in the examined case by a factor of 100) and they decay faster with distance. They also display lower frequency content.

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APPENDIX A

CLASSICAL RAYLEIGH WAVE

We start from the equations of motion

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)\ \mathrm{grad}\ (\mathrm{div}\ \vec{u}) + \mu \Delta \vec{u} \tag{8}$$

and Hooke's law

$$t_{ik} = \lambda \delta_{ik}\ \mathrm{div}\ \vec{u} + \mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right), \quad i,k = 1,2,3, \tag{9}$$

for an isotropic elastic medium.

The stationary solution of Eq. (8) with the time dependence $e^{i\omega t}$ satisfies the following system of equations, if it does not depend on the x_2-coordinate:

$$\begin{aligned}
(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \mu \frac{\partial^2 u_1}{\partial x_3^2} + \rho\omega^2 u_1 &= 0,\\
(\lambda + 2\mu)\frac{\partial^2 u_3}{\partial x_3^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \mu \frac{\partial^2 u_3}{\partial x_1^2} + \rho\omega^2 u_3 &= 0,\\
\mu \frac{\partial^2 u_2}{\partial x_1^2} + \mu \frac{\partial^2 u_2}{\partial x_3^2} + \rho\omega^2 u_2 &= 0.
\end{aligned} \tag{10}$$

We are looking for the vector solution of the system of equations (10), where u_2 vanishes:

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ rA_3 \end{pmatrix} e^{-\omega\alpha x_3} e^{-i\omega p_1 x_1}. \tag{11}$$

This solution represents a wave propagating in x_1-direction with the velocity $1/p_1$. It decays with depth in the halfspace $x_3 \geq 0$. One can show that there is no solution with a non-vanishing u_2-component which satisfies the free surface conditions. Substituting (11) into the system (10) leads to the linear homogeneous system for components A_1 and A_3 of the polarization vector:

$$\begin{aligned}\Big[(\lambda+2\mu)(-)p_1^2+\rho+\mu\alpha^2\Big]A_1+(\lambda+\mu)\alpha i p_1 A_3 &=0,\\ \Big[(\lambda+2\mu)\alpha^2-\mu p_1^2+\rho\Big]A_3+(\lambda+\mu)\alpha i p_1 A_1 &=0.\end{aligned} \tag{12}$$

A non-trivial solution of the system (12) requires that its determinant vanishes. This leads to the vertical slownesses of the P- and S-waves:

$$\alpha_p = \pm\sqrt{p_1^2 - \frac{\rho}{\lambda+2\mu}}, \qquad \alpha_s = \pm\sqrt{p_1^2-\frac{\rho}{\mu}}. \tag{13}$$

The corresponding polarization vectors for P- and SV-waves are

$$\begin{pmatrix} A_1^{(p)} \\ A_3^{(p)} \end{pmatrix} = \begin{pmatrix} p_1 \\ -i\alpha_p \end{pmatrix}$$

and

$$\begin{pmatrix} A_1^{(s)} \\ A_3^{(s)} \end{pmatrix} = \begin{pmatrix} i\alpha_s \\ p_1 \end{pmatrix},$$

respectively.

Therefore, the general solution of the system (Eq. 12) which satisfies the radiation conditions, i.e. the wave decays with the depth or propagates in the positive x_3 direction, can be written in the form

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = C_p \begin{pmatrix} p_1 \\ -i\alpha_p \end{pmatrix} e^{-\omega\alpha_p x_3} e^{-i\omega p_1 x_1} + C_s \begin{pmatrix} i\alpha_s \\ p_1 \end{pmatrix} e^{-\omega\alpha_s x_3} e^{-i\omega p_1 x_1}, \tag{14}$$

where the unknown factors C_p and C_s can be determined from the free surface conditions, i.e. zero stress on the free surface ($x_3 = 0$) of the isotropic elastic halfspace.

According to Hooke's law (Eq. (9)) the components of the stress vector $\vec{T}$ at the surface with the unit normal vector $\vec{i_3}$ can be calculated using:

$$\begin{aligned} T_3 &= t_{33} = (\lambda+2\mu)\frac{\partial u_3}{\partial x_3} + \lambda\frac{\partial u_1}{\partial x_1},\\ T_1 &= t_{31} = \mu\frac{\partial u_3}{\partial x_1} + \mu\frac{\partial u_1}{\partial x_3}.\\ T_2 &= t_{32} = \mu\frac{\partial u_3}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_3}.\end{aligned} \tag{15}$$

If we insert Eq. (14) into the free surface conditions

$$T_1 = t_{31}(x_3=0) = T_3 = t_{33}(x_3=0) = 0,$$

then in order to find non-zero amplitude factors C_p and C_s, we have to solve the algebraic homogeneous system

$$\begin{aligned} i\omega(2\mu p_1^2-\rho)C_p - 2\mu\alpha_s p_1\omega C_s &= 0,\\ -2\alpha_p\omega p_1 C_p + iC_s\omega(\frac{1}{v_s^2}-2p_1^2) &= 0.\end{aligned} \tag{16}$$

A solution only exists if the determinant of the system vanishes:

$$(\frac{1}{v_s^2}-2p_1^2)^2 - 4p_1^2\alpha_p\alpha_s = 0. \tag{17}$$

The real root of Eq. (17) determines the velocity $v_r = \frac{1}{p_{1R}}$ of the Rayleigh wave, and we find the relation

$$C_s = \frac{-i}{2p_1\alpha_s}(\frac{1}{v_s^2} - 2p_1^2)C_p \tag{18}$$

between the components of the polarization vector of the Rayleigh wave. Relation (18) expresses the fact that the Rayleigh wave is elliptically polarized.

It is easy to show that the components of Rayleigh wave (Eq. (14)) with the relation (18) satisfy to the following equation

$$\frac{\partial u_1}{\partial x_3}(x_3 = 0) = u_3(x_3 = 0)i\omega p_1, \tag{19}$$

which will be used in appendix B.

APPENDIX B

RAYLEIGH WAVE WITH COMPONENTS OUT OF THE PLANE OF PROPAGATION

Let us rewrite the Rayleigh solution (Eq. 14) in the following form:

$$\vec{u}^{(R)}(x_1, x_3) = u_1^{(R)}(x_1, x_3)\vec{i}_1 + u_3^{(R)}(x_1, x_3)\vec{i}_3. \tag{20}$$

From Appendix A follows that this solution satisfies the free surface condition for a homogeneous elastic halfspace ($x_3 \geq 0$):

$$t_{33}(\vec{u}^{(R)})(x_3 = 0) = 0, \quad t_{31}(\vec{u}^{(R)})(x_3 = 0) = 0, \quad t_{32}(\vec{u}^{(R)}) \equiv 0,$$

where the components of the stress tensor are calculated by Eqs. (15).

Let us introduce two vector functions

$$\begin{aligned} \vec{u}^{(0)}(x_1, x_2, x_3) &= x_2\vec{u}^{(R)}(x_1, x_3), \\ \vec{u}^{(1)}(x_1, x_3) &= Cu_1^{(R)}(x_1, x_3)\vec{i}_2, \end{aligned} \tag{21}$$

where $u_1^{(R)}(x_1, x_3)$ and $u_3^{(R)}(x_1, x_3)$ are the components of the Rayleigh wave of Eq. (20).

It is easy to show that the vector functions (21) satisfy the equations of motion (Eq. 10). Using formulas (15), the components of the stress vector for these two functions are given by the following equations:

$$\begin{aligned} T_3^{(0)} &= t_{33}(\vec{u}^{(0)}) = x_2 t_{33}(\vec{u}^{(R)}), \\ T_1^{(0)} &= t_{31}(\vec{u}^{(0)}) = x_2 t_{31}(\vec{u}^{(R)}), \\ T_2^{(0)} &= t_{32}(\vec{u}^{(0)}) = \mu u_3^{(R)}, \\ T_3^{(1)} &= t_{33}(\vec{u}^{(1)}) \equiv 0, \\ T_1^{(1)} &= t_{31}(\vec{u}^{(1)}) \equiv 0, \\ T_2^{(1)} &= t_{32}(\vec{u}^{(1)}) = C\mu\frac{\partial u_1^{(R)}}{\partial x_3}. \end{aligned} \tag{22}$$

Therefore, it is evident that the vector function

$$\vec{u} \quad = \vec{u}^{(0)} + \vec{u}^{(1)}$$

satisfies the free surface conditions:

$$\begin{aligned} t_{33}(\vec{u})(x_3 = 0) \quad &= t_{33}(\vec{u}^{(0)})(x_3 = 0) + t_{33}(\vec{u}^{(1)})(x_3 = 0) = 0, \\ t_{31}(\vec{u})(x_3 = 0) \quad &= t_{31}(\vec{u}^{(0)})(x_3 = 0) + t_{31}(\vec{u}^{(1)})(x_3 = 0) = 0, \\ t_{32}(\vec{u})(x_3 = 0) \quad &= t_{32}(\vec{u}^{(0)})(x_3 = 0) + t_{32}(\vec{u}^{(1)})(x_3 = 0) \end{aligned}$$

$$= C\mu \tfrac{\partial u_1^{(R)}}{\partial x_3}(x_3 = 0) + \mu u_3^{(R)}(x_3 = 0).$$

If the constant C in Eqs. (21) and (22) is chosen to be

$$C = \frac{-1}{i\omega p_1},$$

and if we use Eq. (19), the vector function

$$\vec{u}(x_1, x_2, x_3) = x_2 \vec{u}^{(R)}(x_1, x_3) + \frac{-1}{i\omega p_1} u_1^{(R)}(x_1, x_3)\vec{i}_2 \tag{23}$$

satisfies the free surface conditions. From this one can determine the wave which propagates with Rayleigh wave velocity. It has a main component and additional component, which is oriented out of the plane of propagation. The additional component has lower frequency content than the main component.

Formula (23) is the main result of Kiselev (2004). However, this result is of formal character only. It does not describe the real physical problem of elastic wave propagation. To overcome this deficiency we describe the full solution for the wave field due to tangential point source in an elastic halfspace in Appendix C.

APPENDIX C

WAVEFIELD DUE TO A HORIZONTAL FORCE APPLIED TO AN ISOTROPIC HOMOGENEOUS ELASTIC HALFSPACE

If a horizontal force is applied to the elastic halfspace ($x_3 \geq 0$) axial symmetry is no longer given. Therefore, we will solve our problem in Cartesian coordinates (x_1, x_2, x_3). We start from the 3D stationary elastic equations

$$\begin{aligned}
&(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + (\lambda + \mu)(\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}) + \mu(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}) + \rho\omega^2 u_1 = 0,\\
&(\lambda + 2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + (\lambda + \mu)(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_2 \partial x_3}) + \mu(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2}) + \rho\omega^2 u_2 = 0,\\
&(\lambda + 2\mu)\frac{\partial^2 u_3}{\partial x_3^2} + (\lambda + \mu)(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3}) + \mu(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}) + \rho\omega^2 u_3 = 0.
\end{aligned} \tag{24}$$

The source is introduced using boundary conditions for the stress vector at the surface ($x_3 = 0$):

$$\vec{T} = \delta(x_1, x_2)\vec{i_1} \equiv \delta(x, y)\vec{i_1},$$

where the stress vector $T_i = t_{i3}, (i = 1, 2, 3)$ at the surface is given by:

$$\begin{aligned}
T_1 &= \mu(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}),\\
T_2 &= \mu(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}),\\
T_3 &= (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3} + \lambda(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}).
\end{aligned} \tag{25}$$

We need to find the vector solution of the Eq. (24) which satisfies the following conditions at the surface:

$$\begin{aligned}
\mu(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1})(x_3 = 0) &= \delta(x_1, x_2),\\
\mu(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2})(x_3 = 0) &= 0,\\
(\lambda + 2\mu)\frac{\partial u_3}{\partial x_3} + \lambda(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})(x_3 = 0) &= 0.
\end{aligned} \tag{26}$$

We are looking for the solution in the form of a Fourier transform in the horizontal plane (x_1, x_2):

$$u_j(x_1, x_2, x_3) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_j(k_1, k_2, x_3) e^{-i(k_1x_1+k_2x_2)} dk_1 dk_2, \quad j = 1, 2, 3. \tag{27}$$

For a delta function $\delta(x_1, x_2)$ we use a similar representation:

$$\delta(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_1x_1+k_2x_2)} dk_1 dk_2.$$

By using representation (27) we obtain the homogeneous system in the (k_1, k_2, x_3)-space

$$\begin{aligned}
\mu\frac{d^2V_1}{dx_3^2} - ik_1(\lambda+\mu)\frac{dV_3}{dx_3} + [\rho\omega^2 - (\lambda+2\mu)k_1^2 - \mu k_2^2]V_1 - (\lambda+\mu)k_1k_2V_2 = 0,\\
\mu\frac{d^2V_2}{dx_3^2} - ik_2(\lambda+\mu)\frac{dV_3}{dx_3} + [\rho\omega^2 - (\lambda+2\mu)k_2^2 - \mu k_1^2]V_2 - (\lambda+\mu)k_1k_2V_1 = 0,\\
(\lambda+2\mu)\frac{d^2V_3}{dx_3^2} - i(\lambda+\mu)(k_1\frac{dV_1}{dx_3} + k_2\frac{dV_2}{dx_3}) + [\rho\omega^2 - \mu(k_2^2 + k_1^2)]V_3 = 0,
\end{aligned} \tag{28}$$

which, using the substitution

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = e^{\alpha x_3} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \tag{29}$$

results in the following linear algebraic homogeneous system of equations with respect to A_1, A_2, and A_3:

$$\begin{aligned}
[\mu\alpha^2 + \rho\omega^2 - (\lambda+2\mu)k_1^2 - \mu k_2^2]A_1 - (\lambda+\mu)k_1k_2A_2 - ik_1\alpha(\lambda+\mu)A_3 = 0,\\
-(\lambda+\mu)k_1k_2A_1 + [\mu\alpha^2 + \rho\omega^2 - (\lambda+2\mu)k_2^2 - \mu k_1^2]A_2 - ik_2\alpha(\lambda+\mu)A_3 = 0,\\
-i\alpha(\lambda+\mu)(k_1A_1 + k_2A_2) + [(\lambda+2\mu)\alpha^2 + \rho\omega^2 - \mu(k_1^2 + k_2^2)]A_3 = 0.
\end{aligned} \tag{30}$$

Let us construct three independent solutions of system (30) on the basis of plane wave theory by fixing the polarization of the waves.

We consider a SH-wave with the polarization

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \approx \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}. \tag{31}$$

This corresponds to the requirement that the SH-wave is polarized in the horizontal (x_1, x_2)-plane polarized. If Eq. (31) is substituted into Eqs. (30) we can solve for α of Eq. (29) for the SH-wave:

$$\alpha^2 + \frac{\rho}{\mu}\omega^2 = k_1^2 + k_2^2 : \alpha_s = \sqrt{k^2 - \frac{\rho}{\mu}\omega^2} \tag{32}$$

Here and in the following k denotes the horizontal wavenumber $(k_1^2 + k_2^2)^{\frac{1}{2}}$.
We calculate the radical α_s from Eq. (32) on the real axis of the complex plane k by the following radiation condition:

$$\begin{aligned}
\alpha_s = i|\alpha_s| \quad \text{if} \quad -\frac{\omega}{v_s} < k < \frac{\omega}{v_s},\\
\alpha_s = |\alpha_s| \quad \text{if} \quad |k| > \frac{\omega}{v_s}.
\end{aligned}$$

For arbitrary values of k in the complex plane the radical α_s is calculated by analytical continuation from the real axis. The same applies for the calculation of α_p later on.

Let us consider P-SV-waves. We choose the polarization orthogonal to the polarization of the SH-wave such that it is a superposition of two linear independent vectors with unknown coefficients a and b:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = a \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} \tag{33}$$

If representation (33) for the P-SV polarization vectors is put into system (30) the unknown coefficients a and b are a solution of the following system of equations:

$$\begin{aligned} [\mu\alpha^2 + \rho\omega^2 - (\lambda + 2\mu)k^2]b - i\alpha(\lambda + \mu)a = 0, \\ -i\alpha(\lambda + \mu)k^2 b + [\alpha^2(\lambda + 2\mu) + \rho\omega^2 - \mu k^2]a = 0. \end{aligned} \tag{34}$$

Now we consider the SV-wave. If we insert

$$\alpha^2 + \frac{\rho}{\mu}\omega^2 = k^2$$

into system (34), then the polarization of the SV-wave can be chosen as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \alpha_s k_1 \\ \alpha_s k_2 \\ ik^2 \end{pmatrix}.$$

We proceed analogously for the P-wave. If we insert

$$\alpha^2 + \frac{\rho}{\lambda + 2\mu}\omega^2 = k^2 : \alpha_p = \sqrt{k^2 - \frac{\rho}{\lambda + 2\mu}\omega^2}.$$

into system (34), then the polarization of the P-wave can be chosen as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ i\alpha_p \end{pmatrix}.$$

In this way we constructed three independent solutions which satisfy the radiation conditions in the halfspace ($x_3 \geq 0$):

$$e^{-\alpha_p x_3} \begin{pmatrix} k_1 \\ k_2 \\ -i\alpha_p \end{pmatrix}, \qquad e^{-\alpha_s x_3} \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}, \qquad e^{-\alpha_s x_3} \begin{pmatrix} -\alpha_s k_1 \\ -\alpha_s k_2 \\ ik^2 \end{pmatrix}.$$

Finally the general solution of the system (30) is a linear superposition of the form

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = C_1 e^{-\alpha_p x_3} \begin{pmatrix} k_1 \\ k_2 \\ -i\alpha_p \end{pmatrix} + C_2 e^{-\alpha_s x_3} \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} + C_3 e^{-\alpha_s x_3} \begin{pmatrix} -\alpha_s k_1 \\ -\alpha_s k_2 \\ ik^2 \end{pmatrix}. \tag{35}$$

It can easily be verified that conditions (26) are satisfied at the free surface if the integrands of representation (27) fulfil the corresponding conditions

$$\begin{aligned} \mu\Big(\frac{dV_1}{dx_3} - ik_1 V_3\Big)(x_3 = 0) = 1, \\ \mu\Big(\frac{dV_2}{dx_3} - ik_2 V_3\Big)(x_3 = 0) = 0, \\ (\lambda + 2\mu)\frac{dV_3}{dx_3} - i\lambda[k_1 V_1 + k_2 V_2](x_3 = 0) = 0, \end{aligned} \tag{36}$$

which lead to the system

$$\begin{aligned} -2\mu k_1\alpha_p C_1 + k_1 C_3(2\mu k^2 - \rho\omega^2) - C_2\mu k_2\alpha_s &= 1, \\ -2\mu k_2\alpha_p C_1 + \mu k_1\alpha_s C_2 + k_2 C_3(2\mu k^2 - \rho\omega^2) &= 0, \\ C_1(2\mu k^2 - \rho\omega^2) - C_3\alpha_s k^2 2\mu &= 0. \end{aligned} \tag{37}$$

From system (37) we can determine the factors C_1, C_2, and C_3 of the general solution (35):

$$\begin{aligned} C_1 &= \frac{2\mu\alpha_s k_1}{(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s}, \\ C_2 &= -\frac{k_2}{\mu k^2\alpha_s}, \\ C_3 &= \frac{k_1(2\mu k^2 - \rho\omega^2)}{k^2[(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s]}. \end{aligned}$$

Therefore, the components of the wave field in Cartesian coordinates in the k-space are defined by the following expressions:

$$\begin{aligned} V_1 &= \frac{2\mu\alpha_s k_1^2}{(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s} e^{-\alpha_p x_3} - \frac{k_2^2}{\mu k^2\alpha_s} e^{-\alpha_s x_3} \\ &\quad - \frac{k_1^2\alpha_s(2\mu k^2 - \rho\omega^2)}{k^2[(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s]} e^{-\alpha_s x_3}, \\ V_2 &= \frac{2\mu\alpha_s k_1 k_2}{(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s} e^{-\alpha_p x_3} + \frac{k_2 k_1}{\mu k^2\alpha_s} e^{-\alpha_s x_3} \\ &\quad - \frac{k_1 k_2\alpha_s(2\mu k^2 - \rho\omega^2)}{k^2[(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s]} e^{-\alpha_s x_3}, \\ V_3 &= -\frac{2\mu i\alpha_p\alpha_s k_1}{(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s} e^{-\alpha_p x_3} + \frac{ik^2 k_1(2\mu k^2 - \rho\omega^2)}{k^2[(2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s]} e^{-\alpha_s x_3}. \end{aligned} \tag{38}$$

We denote the Rayleigh denominator by

$$R = (2\mu k^2 - \rho\omega^2)^2 - 4\mu^2\alpha_p\alpha_s$$

and introduce the polar coordinates

$$\begin{aligned} x_1 &= r\cos\varphi \qquad x_2 = r\sin\varphi \\ k_1 &= k\cos\theta \qquad k_2 = k\sin\theta \end{aligned}$$

in x- and k-spaces. We calculate the Cartesian components of the displacements as functions of cylindrical coordinates $r, \varphi, z \equiv x_3$ in the form of repeated integrals over the modulus k and angle θ:

$$\begin{aligned}
u_1 &= \frac{1}{(2\pi)^2} \int_0^\infty k^3 dk \frac{2\mu\alpha_s}{R} e^{-\alpha_p x_3} \int_0^{2\pi} \cos^2\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
&- \frac{1}{(2\pi)^2} \int_0^\infty k dk \frac{1}{\mu\alpha_s} e^{-\alpha_s x_3} \int_0^{2\pi} \sin^2\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
&- \frac{1}{(2\pi)^2} \int_0^\infty k dk \frac{\alpha_s(2\mu k^2 - \rho\omega^2)}{R} e^{-\alpha_s x_3} \int_0^{2\pi} \cos^2\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
u_2 &= \frac{1}{(2\pi)^2} \int_0^\infty k^3 dk \frac{2\mu\alpha_s}{R} e^{-\alpha_p x_3} \int_0^{2\pi} \cos\theta\sin\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
&+ \frac{1}{(2\pi)^2} \int_0^\infty k dk \frac{1}{\mu\alpha_s} e^{-\alpha_s x_3} \int_0^{2\pi} \sin\theta\cos\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
&- \frac{1}{(2\pi)^2} \int_0^\infty k dk \frac{\alpha_s(2\mu k^2 - \rho\omega^2)}{R} e^{-\alpha_s x_3} \int_0^{2\pi} \cos\theta\sin\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
u_3 &= -\frac{1}{(2\pi)^2} \int_0^\infty k^2 dk \frac{2\mu i\alpha_p\alpha_s}{R} e^{-\alpha_p x_3} \int_0^{2\pi} \cos\theta e^{-ikr\cos(\theta-\varphi)} d\theta \\
&+ \frac{1}{(2\pi)^2} \int_0^\infty ik^2 dk \frac{2\mu k^2 - \rho\omega^2}{R} e^{-\alpha_s x_3} \int_0^{2\pi} \cos\theta e^{-ikr\cos(\theta-\varphi)} d\theta
\end{aligned} \tag{39}$$

The integration over θ in the integrals over θ and k can be computed analytically. Therefore, only the single integration over k is left in the solution.

Evaluation of special integrals

We intend to evaluate integrals of the form

$$I_1 = \int_0^{2\pi} \sin\theta\cos\theta e^{-ikr\cos(\theta-\varphi)} d\theta.$$

We introduce the new variable $u = \theta - \varphi$. Then

$$I_1 = \cos 2\varphi \int_0^{2\pi} \sin u \cos u e^{-ikr\cos u} du + \sin\varphi\cos\varphi \int_0^{2\pi} (2\cos^2 u - 1) e^{-ikr\cos u} du.$$

The first integral vanishes. Therefore,

$$\begin{aligned}
I_1 &= 2\sin\varphi\cos\varphi \int_0^{2\pi} \cos^2 u e^{-ikr\cos u} du - \sin\varphi\cos\varphi \int_0^{2\pi} e^{-ikr\cos u} du \\
&= -4\pi \sin\varphi\ \cos\varphi\ \frac{d^2 J_0(kr)}{d(kr)^2} - 2\pi \sin\varphi\ \cos\varphi\ J_0(kr).
\end{aligned}$$

In the same way the remaining integrals in (39) are calculated:

$$\begin{aligned}
I_2 &= \int_0^{2\pi} \cos\theta e^{-ikr\cos(\theta-\varphi)} d\theta = -i2\pi\cos\varphi\ J_1(kr), \\
I_3 &= \int_0^{2\pi} \cos^2\theta e^{-ikr\cos(\theta-\varphi)} d\theta = -2\pi\cos^2\varphi\ \frac{d^2 J_0(kr)}{d^2(kr)} + 2\pi\sin^2\varphi\ J_0(kr) + 2\pi\sin^2\varphi\ \frac{d^2 J_0(kr)}{d^2(kr)}, \\
I_4 &= \int_0^{2\pi} \sin^2\theta e^{-ikr\cos(\theta-\varphi)} d\theta = 2\pi\cos^2\varphi J_0(kr) + 2\pi\cos^2\varphi \frac{d^2 J_0(kr)}{d(kr)^2} - 2\pi\sin^2\varphi \frac{d^2 J_0(kr)}{d(kr)^2}.
\end{aligned}$$

Finally we transform the horizontal displacement into polar coordinates using

$$\begin{aligned} u_r &= u_1 \cos\varphi + u_2 \sin\varphi, \\ u_\varphi &= -u_1 \sin\varphi + u_2 \cos\varphi. \end{aligned}$$

Then the components of the wave field are given in cylindrical coordinates:

$$\begin{aligned} u_r &= -\frac{\rho\omega^2 \cos\varphi}{2\pi} \int_0^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} \frac{d^2 J_0(kr)}{d^2(kr)} dk - \frac{\cos\varphi}{2\pi} \int_0^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} [J_0(kr) + \frac{d^2 J_0(kr)}{d^2(kr)}] dk, \\ u_\varphi &= -\frac{\rho\omega^2 \sin\varphi}{2\pi} \int_0^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} [\frac{d^2 J_0(kr)}{d^2(kr)} + J_0(kr)] dk - \frac{\sin\varphi}{2\pi} \int_0^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} \frac{d^2 J_0(kr)}{d^2(kr)} dk, \\ u_3 &= -\frac{\cos\varphi}{2\pi} \int_0^\infty k^2 \frac{2\mu\alpha_p\alpha_s}{R} e^{-\alpha_p x_3} J_1(kr) dk + \frac{\cos\varphi}{2\pi} \int_0^\infty k^2 \frac{2\mu k^2 - \rho\omega^2}{R} e^{-\alpha_s x_3} J_1(kr) dk, \end{aligned} \tag{40}$$

where J_0 and J_1 are 0^{th} and 1^{st} oder Bessel-functions, respectively. In the following we convert half-infinite intervals of integration in the representations (40) into the integration over the whole k-axis. For this purpose we use the relation

$$\frac{d^2 J_0(kr)}{d^2(kr)} + J_0(kr) = \frac{1}{kr} J_1(kr)$$

and rewrite our solution (40) in the following the form:

$$\begin{aligned} u_r &= \frac{\rho\omega^2 \cos\varphi}{2\pi} \int_0^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} J_0(kr) dk - \frac{\rho\omega^2 \cos\varphi}{2\pi} \int_0^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} \frac{1}{kr} J_1(kr) dk \\ &- \frac{\cos\varphi}{2\pi} \int_0^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} \frac{1}{kr} J_1(kr) dk, \\ u_\varphi &= -\frac{\rho\omega^2 \sin\varphi}{2\pi} \int_0^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} \frac{1}{kr} J_1(kr)] dk + \frac{\sin\varphi}{2\pi} \int_0^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} J_0(kr) dk \\ &- \frac{\sin\varphi}{2\pi} \int_0^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} \frac{1}{kr} J_1(kr) dk, \\ u_3 &= -\frac{\cos\varphi}{2\pi} \int_0^\infty k^2 \frac{2\mu\alpha_p\alpha_s}{R} e^{-\alpha_p x_3} J_1(kr) dk + \frac{\cos\varphi}{2\pi} \int_0^\infty k^2 \frac{2\mu k^2 - \rho\omega^2}{R} e^{-\alpha_s x_3} J_1(kr) dk. \end{aligned} \tag{41}$$

For the conversion into a two-sided infinite interval of integration we use the relation

$$J_n(z) = \frac{1}{2} \left[H_n^{(2)}(z) + H_n^{(1)}(z) \right]$$

and formulas for the analytic continuation of Hankel's functions (Abramowitz and Stegun (1968), 9.1.39, p. 361). The result is:

$$\begin{aligned} u_r &= \frac{\rho\omega^2 \cos\varphi}{2\pi} \int_{-\infty}^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} H_0^{(2)}(kr) dk - \frac{\rho\omega^2 \cos\varphi}{2\pi} \int_{-\infty}^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} \frac{1}{kr} H_1^{(2)}(kr) dk \\ &- \frac{\cos\varphi}{2\pi} \int_{-\infty}^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} \frac{1}{kr} H_1^{(2)}(kr) dk, \\ u_\varphi &= -\frac{\rho\omega^2 \sin\varphi}{2\pi} \int_{-\infty}^\infty \frac{k\alpha_s}{R} e^{-\alpha_p x_3} \frac{1}{kr} H_1^{(2)}(kr)] dk + \frac{\sin\varphi}{2\pi} \int_{-\infty}^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} H_0^{(2)}(kr) dk \\ &- \frac{\sin\varphi}{2\pi} \int_{-\infty}^\infty \frac{k}{\mu\alpha_s} e^{-\alpha_s x_3} \frac{1}{kr} H_1^{(2)}(kr) dk, \\ u_3 &= -\frac{\cos\varphi}{2\pi} \int_{-\infty}^\infty k^2 \frac{2\mu\alpha_p\alpha_s}{R} e^{-\alpha_p x_3} H_1^{(2)}(kr) dk + \frac{\cos\varphi}{2\pi} \int_{-\infty}^\infty k^2 \frac{2\mu k^2 - \rho\omega^2}{R} e^{-\alpha_s x_3} H_1^{(2)}(kr) dk. \end{aligned} \tag{42}$$

Finally we change the variable $\omega\zeta = k$ with the slowness ζ and arrive at the final result:

$$
\begin{aligned}
u_r &= \frac{\rho\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}H_0^{(2)}(\omega\zeta r)d\zeta \\
&- \frac{\rho\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta - \frac{\cos\varphi}{2\pi}\int_{0}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta, \\
u_\varphi &= -\frac{\rho\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta\alpha_s}{R}e^{-\omega\alpha_p x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)]d\zeta + \frac{\omega\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}H_0^{(2)}(\omega\zeta r)d\zeta \\
&- \frac{\sin\varphi}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta}{\mu\alpha_s}e^{-\omega\alpha_s x_3}\frac{1}{\zeta r}H_1^{(2)}(\omega\zeta r)d\zeta, \\
u_3 &= -\frac{\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\zeta^2\frac{2\mu\alpha_p\alpha_s}{R}e^{-\omega\alpha_p x_3}H_1^{(2)}(\omega\zeta r)d\zeta \\
&+ \frac{\omega\cos\varphi}{2\pi}\int_{-\infty}^{\infty}\zeta^2\frac{2\mu\zeta^2-\rho}{R}e^{-\omega\alpha_s x_3}H_1^{(2)}(\omega\zeta r)d\zeta.
\end{aligned}
\tag{43}
$$