APPLICATION OF SNELL'S LAW IN WEAKLY ANISOTROPIC MEDIA

C. Vanelle and D. Gajewski

email: *vanelle@dkrz.de*

keywords: *Weak anisotropy, Slowness vector, Snell's law*

ABSTRACT

Computations in anisotropic media are usually cumbersome. One particular problem occurs when two components of the slowness vector are known, yet the full slowness vector is required, e.g., to evaluate Snell's law at an interface between two anisotropic media. In this paper we suggest to combine results from the perturbation method and expressions for sectorially best-fitting isotropic background media in an iterative approach to find the third slowness component. We demonstrate the technique with an example for a medium with triclinic symmetry. The results are in good accordance with the exact solutions.

INTRODUCTION

In modelling and inversion of seismic reflection data the slowness vector

$$\mathbf{p} = \frac{\mathbf{n}}{V} = \frac{1}{V}\,(\sin\theta\,\cos\phi\,,\,\sin\theta\,\sin\phi\,,\,\cos\theta) \quad , \tag{1}$$

plays an important role. In (1) θ and ϕ are the inclination and azimuth angle, respectively, $\mathbf{n}$ is the phase normal, and V is the phase velocity of a compressional or shear wave. We sometimes encounter situations where only two of the slowness components are known, and the third component, e.g., p_3 is searched for. In an isotropic medium, this problem can be easily solved with the eikonal equation,

$$p_i p_i = V^{-2} \quad . \tag{2}$$

In anisotropic media, however, (2) can not generally be solved for p_3 since the phase velocity V depends on the direction of the phase normal $\mathbf{n}$ that also determines the slowness vector. Therefore we need the phase direction to compute the velocity, but we cannot obtain the direction from only two components of the slowness.

One important example for such a situation is the reflection/transmission problem. Here we know the phase direction and velocity of a wave incident on an interface. Snell's law requires that the horizontal slowness, i.e. p_1 and p_2 is preserved. This means that, if the phase velocity of the reflected or transmitted wave is not equal to that of the incident wave, the direction of the phase normal will change accordingly. In the isotropic case we know the phase velocities on both sides of the interface and can thus compute θ and p_3 for the reflected/transmitted wave. If the medium under consideration is, however, anisotropic, we cannot do this, because $V = V(\mathbf{n})$, and $\mathbf{n}$ is unknown for the reflected or transmitted wave. Other examples where we know only two components of the slowness vector occur for the second-order interpolation of traveltimes (Vanelle and Gajewski, 2002), or the determination of geometrical spreading from traveltimes (Vanelle and Gajewski, 2003a). These works are also the basis for the traveltime-based implementation of true-amplitude migration in anisotropic media introduced in Vanelle and Gajewski (2005). In these cases, the horizontal slowness components can be directly obtained from coarsely-gridded traveltime tables, but not the third, vertical component. In isotropic media, we can again evaluate the eikonal equation (2) to solve for p_3, but this does not yield a solution when anisotropy has to be taken into account.

In this paper, we suggest an iterative approach to determine the missing third slowness component based on the first-order perturbation method for anisotropic media (Jech and Pšenčík, 1989). We suggest to apply the perturbation method in combination with the expressions for sectorially best-fitting isotropic background media given in Vanelle and Gajewski (2003b). We will briefly review the basic equations of the perturbation method and weak anisotropy approximation underlying our technique, including an alternative formulation for the weak anisotropy matrix (Pšenčík, 1998). This is followed by the description of our procedure, and the updating of the velocity model. We will then illustrate the technique for qP- and qS-waves on a triclinic medium, and, finally, conclude our results.

METHOD

First-order perturbation method

In the first-order perturbation method the density-normalised elastic parameters of the anisotropic medium, a_{ijkl}, are represented by the sum of the elastic parameters of a suitable background medium, $a^{(0)}_{ijkl}$, and small perturbations Δa_{ijkl}:

$$a_{ijkl} = a^{(0)}_{ijkl} + \Delta a_{ijkl} \quad . \tag{3}$$

Because the perturbations are assumed to be small the first-order perturbation method yields an approximation for weak anisotropy. In this work, we will consider only isotropic background media with the elastic tensor

$$a^{(0)}_{ijkl} = (\alpha^2 - 2\,\beta^2)\,\delta_{ij}\delta_{kl} + \beta^2\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad , \tag{4}$$

where α and β are the isotropic velocities of compressional and shear waves, respectively.

Our aim is the computation of the slowness vector in an anisotropic medium. In terms of the first-order perturbation method, the slowness vector $\mathbf{p}$ is also represented by the sum of the slowness vector in the isotropic background, $\mathbf{p}^{(0)}$, and its perturbation $\Delta\mathbf{p}$. Snell's law stating that the horizontal slowness is preserved requires that the perturbation appears only in the vertical component. Note that we use the term *vertical* in this work in the sense of vertical with respect to a reference surface described by its unit normal vector $\mathbf{z}$. Examples of such reference surfaces are the tangent planes of reflectors or the registration surface. In the latter case the vector $\mathbf{z}$ coincides with the depth direction. In conclusion, we can write the anisotropic slowness vector in the perturbed medium, $\mathbf{p}$, as

$$\mathbf{p} = \mathbf{p}^{(0)} + \boldsymbol{\Delta}\mathbf{p} = \mathbf{p}^{(0)} + \Delta p\,\mathbf{z} \quad , \tag{5}$$

where the scalar quantity Δp is the perturbation of the vertical slowness.

We will apply a result by Jech and Pšenčík (1989) to relate the perturbation of the slowness to the perturbation of the medium parameters. This relation also uses the polarisation vectors $\mathbf{g}^{(0m)}$ in the unperturbed medium. The index m denotes the wavetype, where $m = 3$ is a P-wave and $m = 1, 2$ are S-waves. The polarisation of a P-wave in an isotropic medium equals the phase normal $\mathbf{n}$. The S-wave polarisation vectors are not unique since their only condition is that they are orthogonal to each other and lie in the plane perpendicular to $\mathbf{n}$. However, the degeneration of the S-wave polarisation can be removed with another result from the first-order perturbation method (Jech and Pšenčík, 1989). Consider the two vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$,

$$\mathbf{e}^{(1)} = \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ -\sin\phi \end{pmatrix} \qquad \text{and} \qquad \mathbf{e}^{(2)} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \quad ,$$

that form an orthonormal base with $\mathbf{e}^{(3)} = \mathbf{g}^{(03)} = \mathbf{n}$. Then the polarisation vectors of the S-waves in the unperturbed medium are given by (Jech and Pšenčík, 1989)

$$\begin{aligned} \mathbf{g}^{(01)} &= \quad \mathbf{e}^{(1)}\,\cos\chi + \mathbf{e}^{(2)}\,\sin\chi \\ \mathbf{g}^{(02)} &= -\,\mathbf{e}^{(1)}\,\sin\chi + \mathbf{e}^{(2)}\,\cos\chi \quad . \end{aligned} \tag{6}$$

The angle χ is determined from the weak anisotropy matrix (Pšenčík, 1998)

$$B_{MN} = \Delta a_{ijkl}\, e_i^{(M)}\, e_k^{(N)}\, n_j\, n_l \quad , \tag{7}$$

by

$$\tan 2\chi = \frac{2\,B_{12}}{B_{11} - B_{22}} \quad . \tag{8}$$

Jech and Pšenčík (1989) have shown that the first-order perturbation of the anisotropic eikonal equation,

$$G = 1 = a_{ijkl}\, p_i\, p_l\, g_j\, g_k \quad , \tag{9}$$

leads to

$$\Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} + 2\, a_{ijkl}^{(0)}\, \Delta p_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 0 \quad . \tag{10}$$

Using Equation (5) to express Δp_i, and solving for the perturbation of the vertical slowness results in (Jech and Pšenčík, 1989)

$$\Delta p = -\,\frac{\Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}}{2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}} \quad . \tag{11}$$

Jech and Pšenčík (1989) indicated that Equation (11) can be used iteratively to increase the accuracy of the slowness vector but they do not pursue this any further in that work. However, as we have shown in Appendix A, iterative application of Equation (11) does not converge against the real anisotropic slowness vector, but against its weakly anisotropic approximation. Therefore, repeated application of Equation (11) will only increase the accuracy within the weak anisotropy limit. Also, the manner in which the background model should be updated during the iteration is not discussed by Jech and Pšenčík (1989). In addition to introducing a suitable method for the updating of the background velocities we will demonstrate further below that the iteration can be limited to very few steps, if the initial background velocities are chosen well.

Separation of qP- and qS-waves

As we have already pointed out in the introduction, Equation (11) has the advantage to give us the slowness for a specific wavetype, qP or qS. To apply (11) to either qP- or qS-waves, however, requires both background velocities, α *and* β. In this section we introduce a new formulation of (11) that contains only α for the application to qP-waves, and only β for the qS-waves. Our formulation also includes an alternative expression for the weak anisotropy matrix $\mathbf{B}$ different from (7) to make the expression for shear wave polarisation independent of α.

Substituting Δa_{ijkl} in the numerator of (11) we get

$$\begin{aligned} \Delta a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} &= a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} - a_{ijkl}^{(0)}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} \\ &= a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} - 1 \quad , \end{aligned} \tag{12}$$

where the eikonal equation (9) was applied for the isotropic background medium. For the denominator we use the isotropic elasticity tensor (4) and obtain

$$2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 2\,(\alpha^2 - \beta^2)\, z_i\, g_i^{(0)}\, p_l^{(0)}\, g_l^{(0)} + \beta^2\, z_i\, p_i^{(0)} \quad . \tag{13}$$

In isotropic media, the polarisation vector is $\mathbf{g}^{(03)} = \mathbf{n}$. Furthermore, $\mathbf{p}^{(0)} = \mathbf{n}/\alpha$, and (13) reduces to

$$2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)} = 2\,\alpha^2\, z_i\, p_i^{(0)} \tag{14}$$

for P-waves. For the S-waves it is sufficient to know that the polarisation vectors $\mathbf{g}^{(01)}$ and $\mathbf{g}^{(02)}$ are perpendicular to $\mathbf{n}$ and we obtain

$$2\, a^{(0)}_{ijkl}\, z_i\, p^{(0)}_l\, g^{(0)}_j\, g^{(0)}_k = 2\, \beta^2\, z_i\, p^{(0)}_i \quad . \tag{15}$$

In summary, Equation (11) becomes

$$\Delta p = \frac{1 - a_{ijkl}\, p^{(0)}_i\, p^{(0)}_l\, g^{(0)}_j\, g^{(0)}_k}{2\, V_0^2\, z_i\, p^{(0)}_i} \quad , \tag{16}$$

where V_0 is the isotropic background velocity of the wavetype under consideration, i.e. $V_0 = \alpha$ for qP-waves, and $V_0 = \beta$ for qS-waves.

As $\mathbf{g}^{(03)} = \mathbf{n}$ for P-waves, Equation (16) requires no further knowledge of β in that case. To make (16) independent of α for the application to qS-waves, we need to rewrite the weak anisotropy matrix $\mathbf{B}$ in a fashion that does not contain α any more. This can be achieved by substituting the perturbation of the elastic tensor in (7). With the isotropic elasticity tensor (4), we find that

$$\begin{aligned} a^{(0)}_{ijkl}\, e^{(M)}_i\, e^{(N)}_k\, n_j\, n_l &= (\alpha^2 - 2\,\beta^2)\, e^{(M)}_i\, n_i\, e^{(N)}_k\, n_k + \beta^2\, (e^{(M)}_i\, e^{(N)}_i\, n_k\, n_k + e^{(M)}_i\, n_i\, e^{(N)}_k\, n_k) \\ &= \beta^2\, \delta_{MN} \quad . \end{aligned} \tag{17}$$

Now B_{MN} can be expressed by

$$B_{MN} = a_{ijkl}\, e^{(M)}_i\, e^{(N)}_k\, n_j\, n_l - \beta^2\, \delta_{MN} \quad , \tag{18}$$

which is independent of α. With Equations (16) and (18) we therefore have a new formulation of (11) that requires only the isotropic background velocity of the desired wavetype.

As we have already indicated, the choice of the initial isotropic velocities influences the number of iteration steps of Equation (16) required to obtain the desired accuracy of the slowness vector (within the limitations of the weak anisotropy approximation, see Appendix A). We have found that the expressions for sectorially best-fitting isotropic background velocities derived in Vanelle and Gajewski (2003b) are particularly suited for this application.

We will illustrate our method with an example in the next section where we apply Equation (16) to determine the vertical slowness of qP- and qS-waves in a medium with triclinic symmetry. The example demonstrates that the combination of Equation (16) with the sectorially best-fitting background velocities leads to good results.

EXAMPLE

We have chosen an anisotropic sandstone with triclinic symmetry to demonstrate our method. The density-normalised elastic parameters (in km^2/s^2) were taken from a paper by Mensch and Rasolofosaon (1997):

$$\mathbf{A} = \begin{pmatrix} 4.95 & 0.43 & 0.62 & 0.67 & 0.52 & 0.38 \\ & 5.09 & 1.00 & 0.09 & -0.09 & -0.28 \\ & & 6.77 & 0.00 & -0.24 & -0.48 \\ & & & 2.45 & 0.00 & 0.09 \\ & & & & 2.88 & 0.00 \\ & & & & & 2.35 \end{pmatrix} \quad . \tag{19}$$

We have considered the azimuth angle $\phi = 0$ in this example. We have determined the exact phase velocities and slowness vectors for the qP-, qS$_1$- and qS$_2$-waves from the eigenvalue problem of the Christoffel matrix. These were used as reference. For comparison we have also computed the globally best-fitting isotropic velocities and the velocities in the weak anisotropy approximation introduced by Fedorov (1968).

Finally, we have applied results from Vanelle and Gajewski (2003b) to obtain the sectorially best-fitting velocities for sectors of 30° width in inclination.

To investigate the performance of Equation (16) we have considered the case that the unit normal to the reference surface, $\mathbf{z}$ coincides with the vertical direction (depth), i.e. $\mathbf{z} = (0, 0, 1)$. This means that we know the horizontal slowness components p_1 and p_2 and wish to determine the vertical slowness p_3 with Equation (16). The vertical slowness in the isotropic background can be computed from the isotropic eikonal equation by

$$p_3^{(0)} = \pm\sqrt{\frac{1}{V_0^2} - p_1^2 - p_2^2} \quad , \tag{20}$$

where the sign of p_3 is chosen according to whether we consider an incident or emerging wave, or a reflected or transmitted wave. After the first iteration step, the background velocity for the next step is updated using

$$\frac{1}{V_0^2} = p_1^2 + p_2^2 + p_3^2 \quad , \tag{21}$$

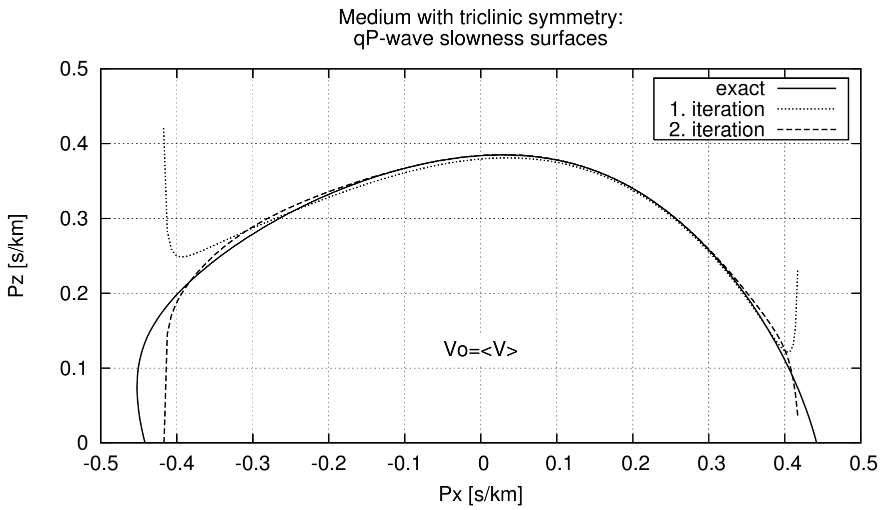
and Equation (16) is applied again.

We have carried out the first two iteration steps for qP- and qS-waves with two sets of initial background velocities. As we have shown in the previous paragraph, the sectorially best-fitting isotropic velocities provide a good fit to the real slowness surface, and we would expect that only few iteration steps are required. This expectation is confirmed, as Figures 1a and 2a prove for the qP- and qS-waves, respectively. Closer inspection shows that at larger inclination angles the vertical slowness is not well-behaved. The reason is that the denominator in Equation (16) vanishes for $p_3^{(0)} = 0$. We can observe that already small values of $p_3^{(0)}$ can lead to large correction terms Δp and make the algorithm unstable. Too small values of $p_3^{(0)}$ can also occur for larger horizontal slownesses when the background velocity is higher than the anisotropic phase velocity. It is even possible that $p_3^{(0)}$ becomes imaginary if the inverse of the background velocity is smaller than the horizontal slowness. For these phase directions, the initial velocity must be chosen smaller.

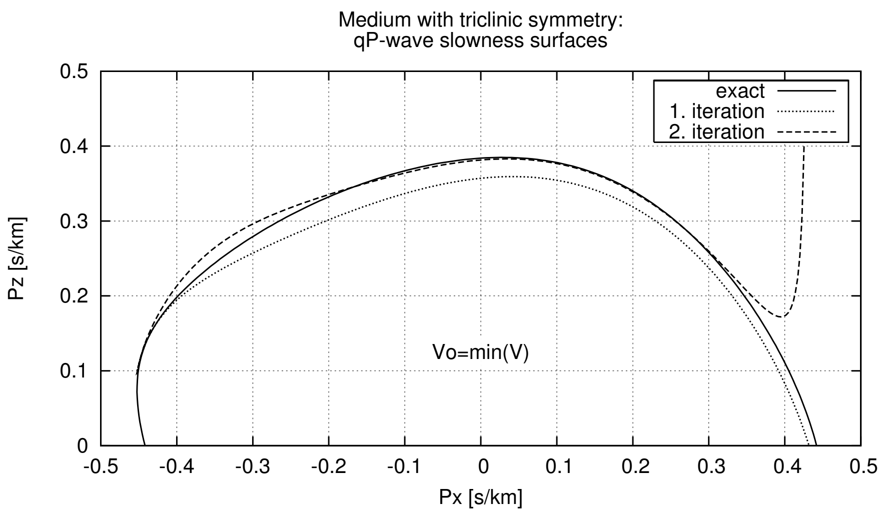
We have, therefore, repeated our experiment with initial background velocities that are just below the minimum phase velocities in the anisotropic medium. These values could be determined in our example because we have computed the exact slowness vectors, and thus the phase velocities, as reference (see also Vanelle and Gajewski (2003b)). The background phase velocity chosen for the qP-wave was 2.15 km/s, and that for the qS-waves was 1.35 km/s. The results from the first two iteration steps are displayed in Figures 1b and 2b for the qP- and qS-waves, respectively.

Our first observation in Figures 1 and 2 is that the vertical slowness after the first step is much worse for these minimum velocities. Although the second iteration improves the result, it is still less good than the first iteration using the sectorially best-fitting velocities. Particularly for the faster $q\mathrm{S}_1$-wave the second iteration does not yield a satisfactory result. Taking into account that after the first iteration the problem with small or imaginary values of $p_3^{(0)}$ occurs also when the initial background velocities were smaller than the anisotropic phase velocities, we strongly recommend the sectorially best-fitting background velocities for the initial step. This commendation is supported by the fact that high inclination angles correspond to post-critical reflections or very large incidence/emergence angles at seismic sources or receivers. These situations are not of high interest within reflection seismics. For the remaining lower incidence angles our method provides good results.

Finally, we can observe that there are regions in which the accuracy of the method is lower. This is, for example, the case for $p_1 \approx -0.3$km/s for the qP-wave and $p_1 \approx -0.4$km/s for the qS-waves, where we find that the weak anisotropy approximation shows larger deviations from the exact results. Our method converges against the weakly anisotropic slowness vector and this leads to errors in these regions. This is further illustrated by Figure 3, where we have plotted the results from the second iteration using sectorially best-fitting background velocities together with the weak anisotropy slowness surfaces.

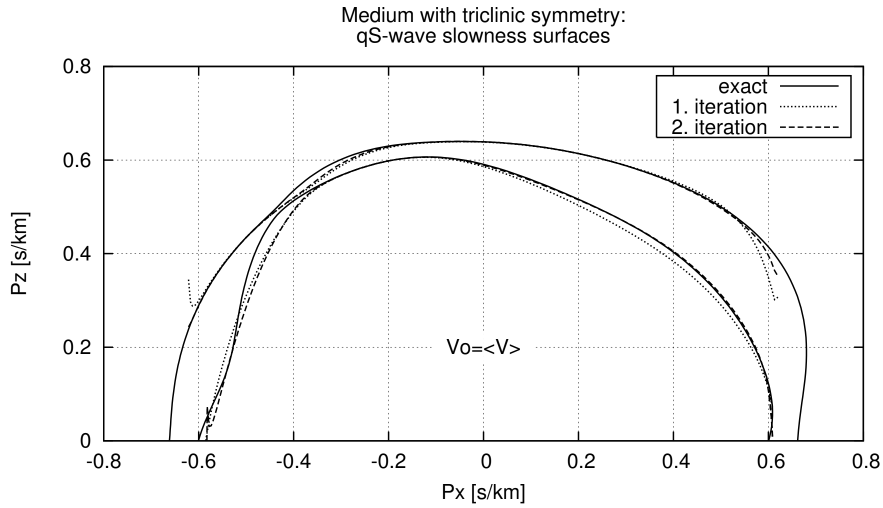


(a) Slowness surfaces of qP-waves with sectorially best-fitting isotropic velocity as initial background model.

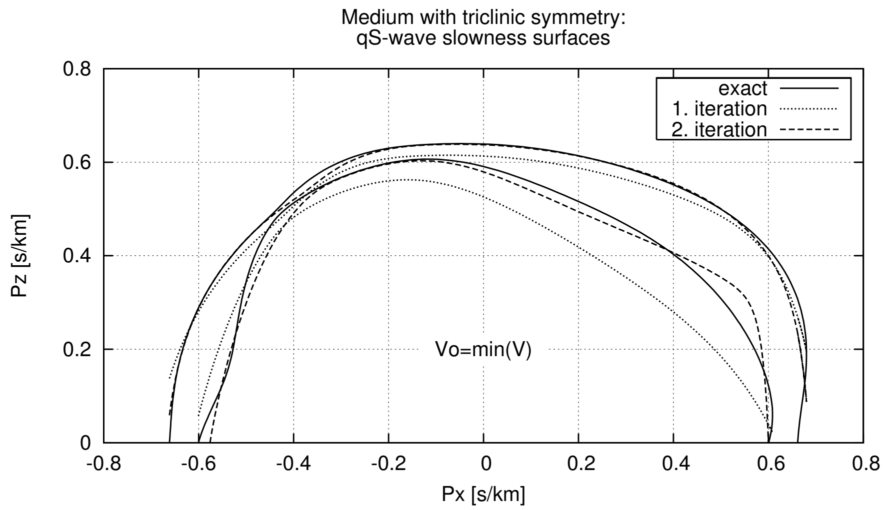


(b) Slowness surfaces of qP-waves with minimum velocity as initial background model.

Figure 1: Slowness surface plots of the exact slowness for qP-waves and values obtained after the first and second iteration from the vertical component. In (a) the sectorially best-fitting isotropic P-wave was chosen as initial background velocity. The approximation is very close to the exact value after the second iteration. In (b) the minimal velocity was chosen as initial background velocity and the result after the second iteration step is less close than for (a), especially at $p_x = -0.35$ s/km. The effects towards lower vertical slowness values are discussed in the text.

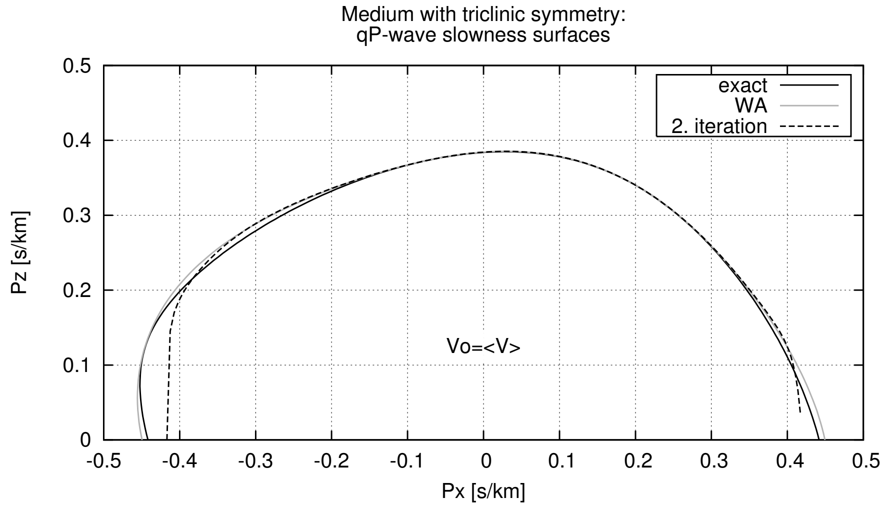


(a) Slowness surfaces of qS-waves with sectorially best-fitting isotropic velocity as initial background model.

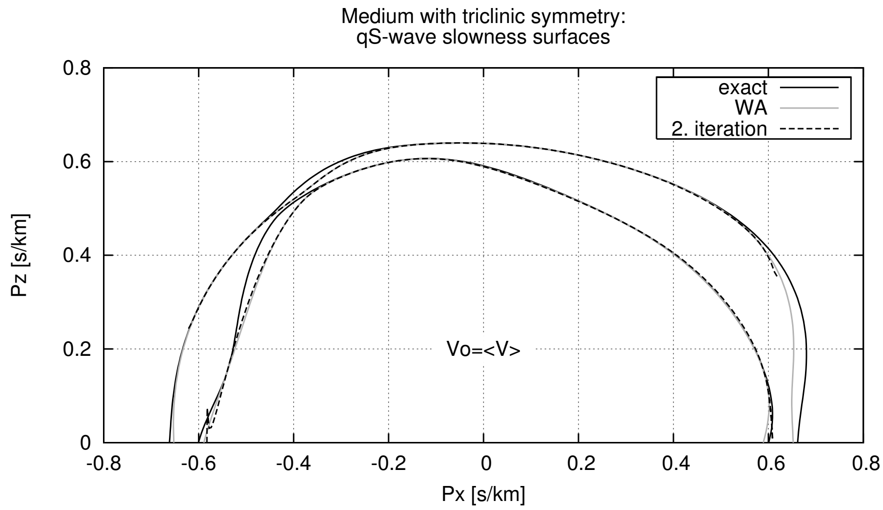


(b) Slowness surfaces of qS-waves with minimum velocity as initial background model.

Figure 2: Slowness surface plots of the exact slowness for qS-waves and values obtained after the first and second iteration from the vertical component. In (a) the sectorially best-fitting isotropic S-wave was chosen as initial background velocity. The approximation is very close to the exact value after the second iteration. In (b) the minimal velocity was chosen as initial background velocity and the result after the second iteration step is less close than for (a), especially for the fast wave. The effects towards lower vertical slowness values and the misfit around $p_x = -0.5$ s/km are discussed in the text.



(a) Slowness surfaces of qP-waves: convergence against the weak anisotropy approximation.



(b) Slowness surfaces of qS-waves: convergence against the weak anisotropy approximation.

Figure 3: Convergence of the iteration against the weakly anisotropic slowness vector for qP-waves (a), and qS-waves (b). The results of the second iteration with the best-fitting isotropic velocities as initial background model are displayed together with the exact results and the weakly anisotropic slowness surfaces.

CONCLUSIONS

We have applied a combination of the first-order perturbation method and expressions for sectorially best-fitting isotropic background media in an iterative approach to determine the vertical component of the slowness vector in arbitrarily anisotropic media. Numerical examples on a medium with triclinic symmetry show that good accuracy is already achieved after the first or second iteration. The method is also applicable to shear waves, however, the accuracy is lower in the vicinity of shear wave singularities. The main applications for the method are the reflection-transmission problem between two anisotropic media, and the traveltime-based determination of geometrical spreading and true-amplitude migration weight functions for anisotropic media.

ACKNOWLEDGEMENTS

We thank the members of the Applied Geophysics Groups in Hamburg and Ivan Pšenčík for continuous discussions. This work was partially supported by the sponsors of the Wave Inversion Technology (WIT) Consortium and the German Research Foundation (DFG, grants Ga 350/11 and Va 207/2).

REFERENCES

Fedorov, F. I. (1968). *Theory of Elastic Waves in Crystals*. Plenum Press, New York.

Jech, J. and Pšenčík, I. (1989). First-order perturbation method for anisotropic media. *Geophysical Journal International*, 99:369–376.

Mensch, T. and Rasolofosaon, P. N. J. (1997). Elastic wave velocities in anisotropic media of arbitrary anisotropy – generalisation of Thomsen's parameters ϵ, δ, and γ. *Geophysical Journal International*, 128:43–63.

Pšenčík, I. (1998). Green's functions for inhomogeneous weakly anisotropic media. *Geophysical Journal International*, 135:279–288.

Vanelle, C. and Gajewski, D. (2002). Second-order interpolation of traveltimes. *Geophysical Prospecting*, 50:73–83.

Vanelle, C. and Gajewski, D. (2003a). Determination of geometrical spreading from traveltimes. *Journal of Applied Geophysics*, 54:391–400.

Vanelle, C. and Gajewski, D. (2003b). Determination of sectorially best-fitting isotropic background media. *7th Annual WIT Report*.

Vanelle, C. and Gajewski, D. (2005). An introduction to true-amplitude migration in anisotropic media. *9th Annual WIT Report*.

APPENDIX A – CONVERGENCE AGAINST THE WEAK ANISOTROPY APPROXIMATION

Let us assume that the iterative application of Equation (11) leads to the anisotropic slowness vector $\mathbf{p}$. This would mean that, after convergence has been achieved, we would expect Δp to become zero in further iteration steps, as $\mathbf{p}$ and $\mathbf{p}^{(0)}$ become equal. Substituting the anisotropic eikonal equation (9) into (11) we find that

$$\Delta p = \frac{a_{ijkl}\, p_i^{(0)}\, p_l^{(0)}\, (g_j\, g_k - g_j^{(0)}\, g_k^{(0)})}{2\, a_{ijkl}^{(0)}\, z_i\, p_l^{(0)}\, g_j^{(0)}\, g_k^{(0)}} \quad . \tag{22}$$

Since generally in an anisotropic medium $g_j \neq g_j^{(0)}$, Δp will not vanish, meaning that the iteration procedure does not converge against the anisotropic slowness vector. For a weakly anisotropic medium, however, the polarisation vectors coincide with those in the isotropic background introduced above, and Δp becomes zero. Therefore, the iterative use of (11) converges against the slowness vector in the weak anisotropy approximation.