EFFECTIVE CONDUCTIVITY AND DIFFUSIVITY OF RANDOMLY HETEROGENEOUS POROUS SOLIDS WITH COMPRESSIBLE CONSTITUENTS

Tobias M. Müller

email: *tobias.mueller@gpi.uni-karlsruhe.de*
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ABSTRACT

Inhomogeneous fluid-saturated porous materials are often probed with diffusion waves to estimate their effective conductivity and diffusivity. Analysis of diffusion wave fields in randomly inhomogeneous poroelastic structures provides new insight how fluctuations of the compressible constituents affect the effective diffusivity. Based on the method of statistical smoothing an effective wave number of the coherent diffusion wave field is computed. From this wave number both an effective conductivity and diffusivity are identified. The correspondence between this conductivity and that estimated from unsteady flow through porous media based on Darcy's law is elucidated. It is shown that in the limits of low and high frequencies these effective conductivities are identical.

INTRODUCTION

Probing complex structures with diffusion waves has become a powerful technique in various physical disciplines (Mandelis, 2001). In particular, diffusion waves are used to characterize transport properties of fluid-saturated porous solids (Rice and Cleary, 1976). If the porous material has a deformable frame, the transport properties can be affected by the compressibility of the solid as well as fluid phase. This effect can be analyzed using Biot's theory of poroelasticity (Biot, 1962). In particular, Biot's theory predicts the existence of so-called Biot's slow wave, which in the low-frequency (i.e., quasi-static) limit is governed by the diffusion equation (Chandler and Johnson, 1981). The corresponding diffusion coefficient depends on compressibilities of the fluid and solid phases as well as permeability and fluid viscosity.

Transport properties of porous materials are particularly affected by spatial heterogeneity. The study of these effects usually employs the concept of random media, and requires an analysis of field equations with random coefficients. One method that can be applied in this context is method of statistical smoothing (Karal and Keller, 1964), which has been widely used in the analysis of wave propagation in random media as well as flow through random rigid porous media (King, 1987; Keller, 2001).

In this letter we will employ the method of statistical smoothing to compute the effective diffusivity of randomly inhomogeneous porous media. The medium is assumed to be governed by the low-frequency version of Biot's equations of poroelasticity where the slow compressional wave is a diffusion wave characterized by the wave number

$$k_0 = \sqrt{i\omega/D_0} \qquad (1)$$

with diffusivity D_0 (Norris, 1985). Neglecting the interaction with other wave modes, we analyze the coherent diffusion wave field only. We derive an expression for the effective diffusion wave number from which we extract the effective transport properties.

STATISTICAL SMOOTHING IN POROELASTIC MEDIA

In random poroelastic media all parameters are represented by random fields of the form $X = \bar{X} + \tilde{X} = \bar{X}(1+\varepsilon_X)$, where $\bar{X}$ is a constant background value and $\tilde{X}(\mathbf{r})$ is the fluctuating part. Parameter $\varepsilon_X = \tilde{X}/\bar{X}$ denotes the relative fluctuations and has zero mean ($\langle\varepsilon_X\rangle = 0$), autocorrelation function $B_{XX}(\delta\mathbf{r}) = \langle\varepsilon_X(\mathbf{r}+\delta\mathbf{r})\varepsilon_X(\mathbf{r})\rangle$, and variance $\langle\varepsilon_X^2\rangle = B_{XX}(0) = \sigma_{XX}^2$. The starting point of our analysis is the poroelastic Dyson integral equation for the mean Green's function of Biot's equations (Müller and Gurevich, 2005) in the random porous medium. By neglecting all contributions from the fast compressional and shear waves, we can write Dyson's equation for the matrix containing the mean Green's tensors $\bar{\mathbf{G}}$ in the form

$$\bar{\mathbf{G}} = \mathbf{G}_0 + \int\int \mathbf{G}_0\mathbf{Q}\bar{\mathbf{G}}\,, \tag{2}$$

where $\mathbf{G_0}$ denotes the matrix of Green's function for the homogeneous background (Müller and Gurevich, 2005)

$$\mathbf{G_0} = \frac{\kappa_0}{4\pi i\omega}\partial_i\partial_j\frac{\exp(ik_0R)}{R}\begin{bmatrix} -\frac{C^2}{H^2} & \frac{C}{H} \\ \frac{C}{H} & -1 \end{bmatrix}, \tag{3}$$

where R denotes the distance from source to observation point and ∂_i denotes partial spatial derivative. $\mathbf{Q}$ is the matrix of the kernel-of-mass operators

$$\mathbf{Q} = \left\langle \tilde{\mathbf{L}}\mathbf{G}_0\tilde{\mathbf{L}} + \int \tilde{\mathbf{L}}\mathbf{G}_0\tilde{\mathbf{L}}\mathbf{G}_0\tilde{\mathbf{L}} + \int ... \right\rangle, \tag{4}$$

where $\tilde{\mathbf{L}}$ denotes the matrix of the perturbing operators

$$\tilde{\mathbf{L}} = \begin{bmatrix} \partial_i\tilde{H}\partial_j & \partial_i\tilde{C}\partial_j \\ \partial_i\tilde{C}\partial_j & i\omega\tilde{p}\delta_{ij} + \partial_i\tilde{M}\partial_j \end{bmatrix} \tag{5}$$

with the identity tensor δ_{ij}. The method of statistical smoothing consists now in truncating $\mathbf{Q}$ after the first term. In Eqs. (3) and (5) H is the undrained, low-frequency P-wave modulus given by Gassmann's equation $H = P_d + \alpha^2 M$ where M is the pore space modulus $M = [(\alpha-\phi)/K_g + \phi/K_f]^{-1}$ and $P_d = K_d + 4/3\mu$ is the P-wave modulus of the drained frame, $\alpha = 1 - K_d/K_g$ is the Biot-Willis coefficient, $C = \alpha M$. K_g, K_d, and K_f denote the bulk moduli of the solid phase, the drained frame, and the fluid phase, while μ denotes the porous-material shear modulus. In (3) the conductivity is denoted as κ_0 while in (5) $\tilde{p}$ denotes the fluctuating part of the reciprocal conductivity $p = 1/\kappa_0$. The diffusivity in (1) can be expressed through $D_0 = \kappa_0 N$ where $N = MP_d/H$.

Eq. (2) contains a double convolution which in the spatial Fourier domain yields a set of algebraic equations. Retaining only terms of order $O(\epsilon^2)$ a simpler equation for the $[2,2]$ component of $\bar{\mathbf{G}}$ is obtained:

$$\bar{g} = g_0 + (8\pi^3)^2 g_0 q\bar{g}\,, \tag{6}$$

where g_0 and and q are the Fourier transforms of the corresponding components of $\mathbf{G_0}$ and $\mathbf{Q}$, respectively. Assuming that the mean Green's function $\bar{g}$ is of the same functional form as g_0, but involving an effective wave number k^*, we can solve equation (6) for k^*. The truncated kernel-of-mass operator matrix element q can be evaluated for statistically isotropic random media and yields the following approximation for the square of the effective wave number

$$k^{*2} = k_0^2[1 + \Delta_s\xi(\omega)]\,, \tag{7}$$

where

$$\xi(\omega) = 1 + k_0^2\int_0^\infty rB(r)\exp(ik_0r)dr \tag{8}$$

and

$$\Delta_s = \left\langle\left(\frac{\alpha^2M}{P_d}\varepsilon_\alpha - \varepsilon_{K_f} + \varepsilon_\phi\right)^2\right\rangle + \frac{\sigma_{pp}^2}{3}\,. \tag{9}$$

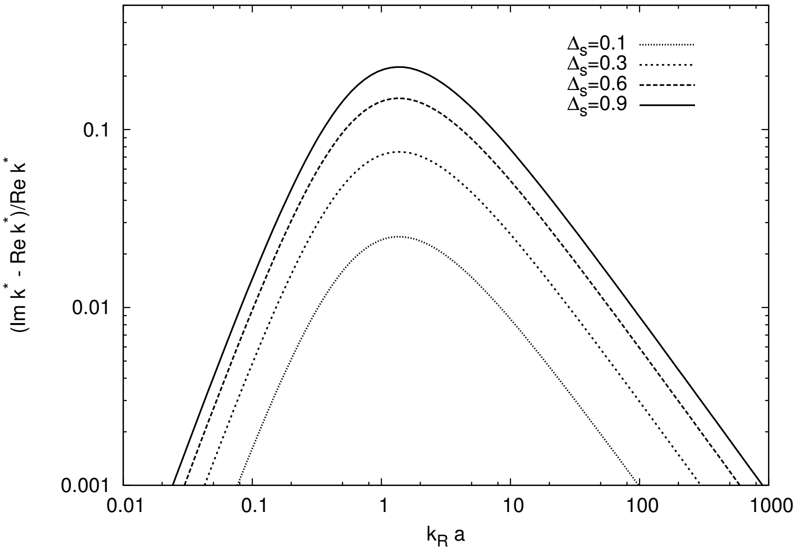

Figure 1: Normalized difference of real and imaginary part of the effective wave number versus frequency.

The method of statistical smoothing is precise for weak fluctuations only, and therefore, Eq. (7) is applicable if $\Delta_s < 1$. The physical interpretation of the effective diffusion wave number is straightforward. Due to multiple scattering, or in terms of diffusion wave terminology (Mandelis, 2001), due to accumulation and depletion processes at randomly spaced inhomogeneities, an initially homogeneous diffusion wave with k_0 becomes at finite frequencies an inhomogeneous diffusion wave characterized by k^* (i.e., $\Re\{k^*\} \neq \Im\{k^*\}$). In the limits of zero and infinite frequency the diffusion wave becomes homogeneous again with the effective wave numbers $k^*(\omega \to 0) = k_0(1+\Delta_s)$ and $k^*(\omega \to \infty) = k_0$, respectively. The frequency dependence of this phenomenon is illustrated in Figure 1.

EFFECTIVE DIFFUSIVITY AND CONDUCTIVITY

Effective wave number (7) can be used to construct approximation of the effective transport properties. Assuming that the effective wave number k^* involves an effective diffusivity D^* such that $k^* = \sqrt{i\omega/D^*}$, we obtain

$$D^* = D_0[1 - \Delta_s \xi(\omega)]\,. \tag{10}$$

Figure 2 illustrates the frequency dependence of the effective diffusivity for varying Δ_s in a model of a porous sandstone where the inhomogeneities are statistically characterized by a correlation function of the form $B(r) = \exp(-|r|/a)$ where a is the correlation length [shown is the real part of D^* as a function of the dimensionless frequency $k_R a$ where k_R denotes the real part of k_0]. As can be seen from Figure 2, the presence of inhomogeneities reduces the effective diffusivity below the background diffusivity. In the zero frequency limit we have $D^*(\omega \to 0) = D_0(1 - \Delta_s)$, whereas for infinite high frequency the background value is obtained, $D^*(\omega \to \infty) = D_0$. The magnitude of the diffusivity dispersion is controlled by Δ_s which contains the second order moments of the random fields of α, ϕ, K_f and p. The role of cross-correlations is particularly interesting. For example, negative cross-correlation between the Biot-Willis coefficient and the fluid bulk modulus, i.e. if there is a stiff fluid in the pore-space of a very compressible porous solid, produces an enhanced diffusivity dispersion.

Analogously to D^*, we can construct an effective conductivity κ^* by assuming that k^* is of the form $k^* = \sqrt{i\omega/\kappa^* N}$, i.e., N is constant. In this case the calculations outlined above can be performed in all space dimensions and we obtain

$$\kappa^* = \kappa_0 \left[1 - \frac{\sigma_{pp}^2}{m}\left(1 - 4\int \frac{k_R^4}{4k_R^4 + K^4}\Phi(K)d\mathbf{K}\right)\right] \tag{11}$$

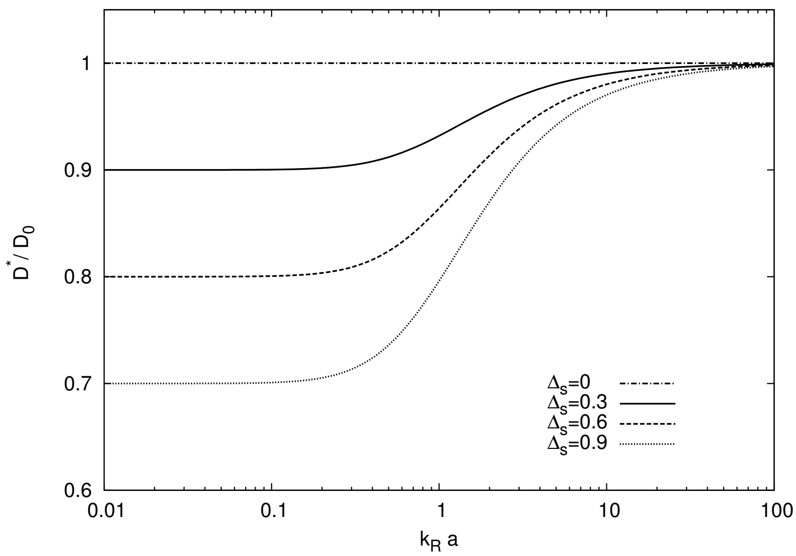

Figure 2: Normalized effective diffusivity versus frequency for a porous sandstone model.

in m-dimensional space (m=1, 2 or 3) where $\Phi(K)$ denotes the fluctuation spectrum. This result is displayed in Figure 3. The effective conductivity is bounded by

$$\kappa_H \leq \kappa^* \leq \kappa_A \, , \tag{12}$$

where κ_H and κ_A denote the harmonic and arithmetic averages, respectively. In the low-frequency limit the lower bound becomes an identity for $m = 1$ only, whereas at infinitely high frequencies the upper bound is reached exactly in all space dimensions.

RELATION TO UNSTEADY FLOW IN POROUS MEDIA

For a poroelastic continuum the diffusion wave mode (Biot's slow wave) in the limit of zero frequency is equivalent to the quasi-static flow (Chandler and Johnson, 1981). It is therefore interesting to compare our results with estimates of an effective conductivity in random porous media based on Darcy's law (Hristopulos and Christakos, 1997; Keller, 2001). Time-dependent, i.e. unsteady, flow analyzed on the basis of Darcy's law in conjunction with the time-dependent continuity equation yields a diffusion equation for the pore pressure of the form $S\partial_t P = \nabla \cdot (\kappa \nabla P)$, where S is the specific storativity (Dagan, 1982; Indelmann, 1996). In statistically isotropic m-dimensional random media an averaged Darcy law allows to define an effective conductivity. The latter can be explicitly computed for weakly inhomogeneous structures involving only the spatial correlation of the conductivity fluctuations (Indelmann, 1996):

$$\kappa^*_{flow} = \kappa_0 \left[1 - \frac{\sigma^2_{\kappa\kappa}}{m} \int \frac{K^2}{k_R^2 + K^2} \Phi(K) d\mathbf{K} \right] \, . \tag{13}$$

This effective conductivity is also displayed in Figure 3. Note that the low and high frequency limits as well as the inflection point of κ^* and κ^*_{flow} [Eqs. (11) and (13)] are identical. Also, both approaches show that in the weak fluctuation case the effective conductivity does not depend on the compressibilities of the porous material and the fluid phase.

CONCLUSIONS

In conclusion, the main result of this letter is the expression for the effective diffusivity (10). It depends not only on the the second-order statistics of the conductivity but also on that of the poroelastic moduli. Therefore, for accurate estimation of the effective conductivity from diffusion wave characteristics, fluctuations of the compressibilities of the porous material must be accounted for.

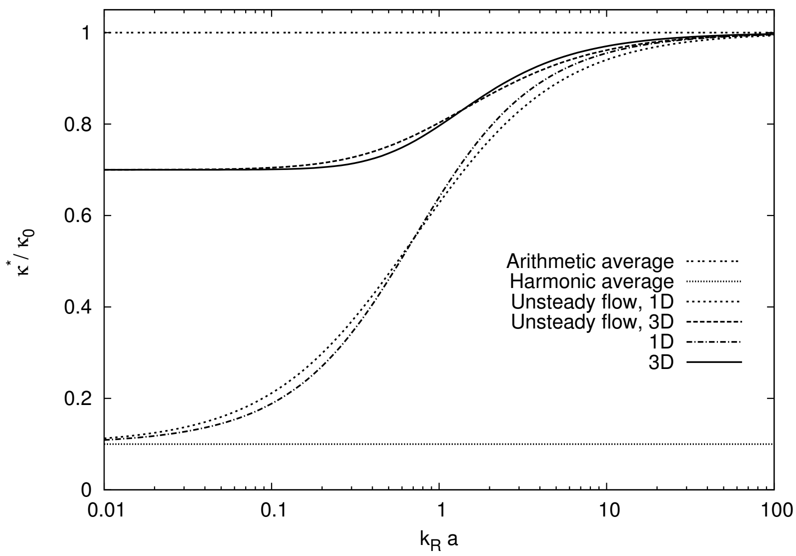

Figure 3: Normalized effective conductivity versus frequency.

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