INVESTIGATION OF THE ONE-WAY WAVE EQUATION

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ABSTRACT

The acoustic one-way wave equation has applications in the fields of reverse time migration and reverse modelling for localization. Generally it is desired that not only the kinematics is correctly reproduced by the one-way wave equation but also amplitudes need to be correct. We examine the respective equations and find that the commonly used solution in modelling for the one-way wave equation is dynamically not exact since it is lacking a wavenumber factor k. In a numerical example for a homogeneous model we found an amplitude error of about 50 %. Moreover, it is theoretically shown, that the one-way wave equation produces artifacts, which are also observable in numerically modelling. These artifacts manifest themselves in 2-dimensional modelling in form of plane waves travelling in the horizontal direction. These waves transport non-negligible amplitudes.

INTRODUCTION

The acoustic one-way wave equation is of importance in reverse time migration, both in post-stack, e.g. Baysal et al. (1983) and in pre-stack case. It is also applied in localization techniques based on time reversed modelling, e.g. Gajewski and Tessmer (2005). Usually the one-way wave equation is designed such that wave propagation in the horizontal (x_1-)direction is allowed both, in $+x_1$- and $-x_1$-direction. In the vertical (x_2-)direction, however, wave propagation is usually only allowed downward, i.e. in the $+x_2$-direction. This makes sure that multiple reflections between horizontal layer boundaries cannot occur.

In the following we limit ourselves to the 2D case for the sake of simplicity. However, the considerations apply as well to the 3D case. In numerical modelling results it was observed that artifacts are present when using the one-way wave equation. These artifacts appear as plane wave fronts which are oriented perpendicular to the horizontal axis. These wave fronts travel exactly in the horizontal direction.

Analysis of the one-way wave equation revealed that the analytic solution contains contributions which explain the above mentioned artifacts.

ANALYTIC WAVE FIELDS USING THE FOURIER TRANSFORM

Theoretical wave field solutions at a given time were computed in the wave number domain and subsequently Fourier transformed into the space domain. This allows direct comparisons of results from the two-way and the correct and incorrect one-way wave equations. Details of the theory for computing the one-way and two-way wave fields are given in the Appendix.

To show the correctness of the formulation of the acoustic 2D one-way wave equation solutions of the two-way and the one-way wave equations are compared.

For the computations of the two-way wave equation Eq. (5) and for the correct one-way wave equation Eq. (12) were used. The medium has constant velocity of 1000 m/s, the propagation time is 175 ms, the spatial discretization is 1 m and the parameter a which determines the spatial width of the initial pulse is equal to one.

the two-way and the one-way wave equations. At the top the wave fields are shown. The left and right sides show results of the two-way and the one-way wave equation, respectively. Below the wave field displays horizontal sections through the wave field at depths of 350 m, 400 m, and 450 m are displayed. The displays of the results of the one-way wave equation show the above mentioned artifacts, i.e. the horizontally travelling plane wave front. This can be seen very clearly in the bottom section at 450 m at which the wave front has not yet arrived. Note the amplitude scale.

For better quantitative comparison Fig. 2 shows horizontal sections of the one-way and two-way wave equation at different depths merged together. As can be seen from the plots the agreement is, apart from the artifacts, very good.

A difference plot of the wave fields of the two-way and the one-way wave equations is given in Fig. 3. The lower semicircle of the wave front vanishes. This shows perfect agreement of the wave fields of both wave equations in their downward travelling parts.

In Fig. 4 the solutions of the wrong and the correct one-way wave equations are shown in juxtaposition. A considerable difference of amplitudes can be observed. In the solution of the incorrect wave equation amplitudes appear about 50% too large compared to the correct one-way wave equation.

CONCLUSIONS

We have theoretically shown that the acoustic one-way wave equation needs an additional wavenumber factor (k) in the denominator to be correct. Otherwise amplitude errors occur. This was also verified by direct comparison with results of the two-way wave equation. It is expected that similar amplitude errors will also arise in inhomogeneous media. However, we cannot prove this since we have no analytic solution for this case. To date we have not found a way to improve the one-way wave equation in numerical modelling like it was done for the homogeneous analytical situation. Furthermore, the plane wave like artifacts which occur when modelling with the one-way wave equation can be explained theoretically.

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REFERENCES

Baysal, E., Kosloff, D., and Sherwood, J. W. C. (1983). Reverse-time migration. *Geophysics*, 48:1514–1524.

Bleistein, N. (1984). *Mathematical methods for wave phenomena*. Academic Press.

Gajewski, D. and Tessmer, E. (2005). Reverse modelling for seismic event characterization. *GJI*, 163:276–284.

Gradsteyn, I. S. and Ryzhin, I. M. (1965). *Table of integrals, series and products*. Academic Press, New York & London.

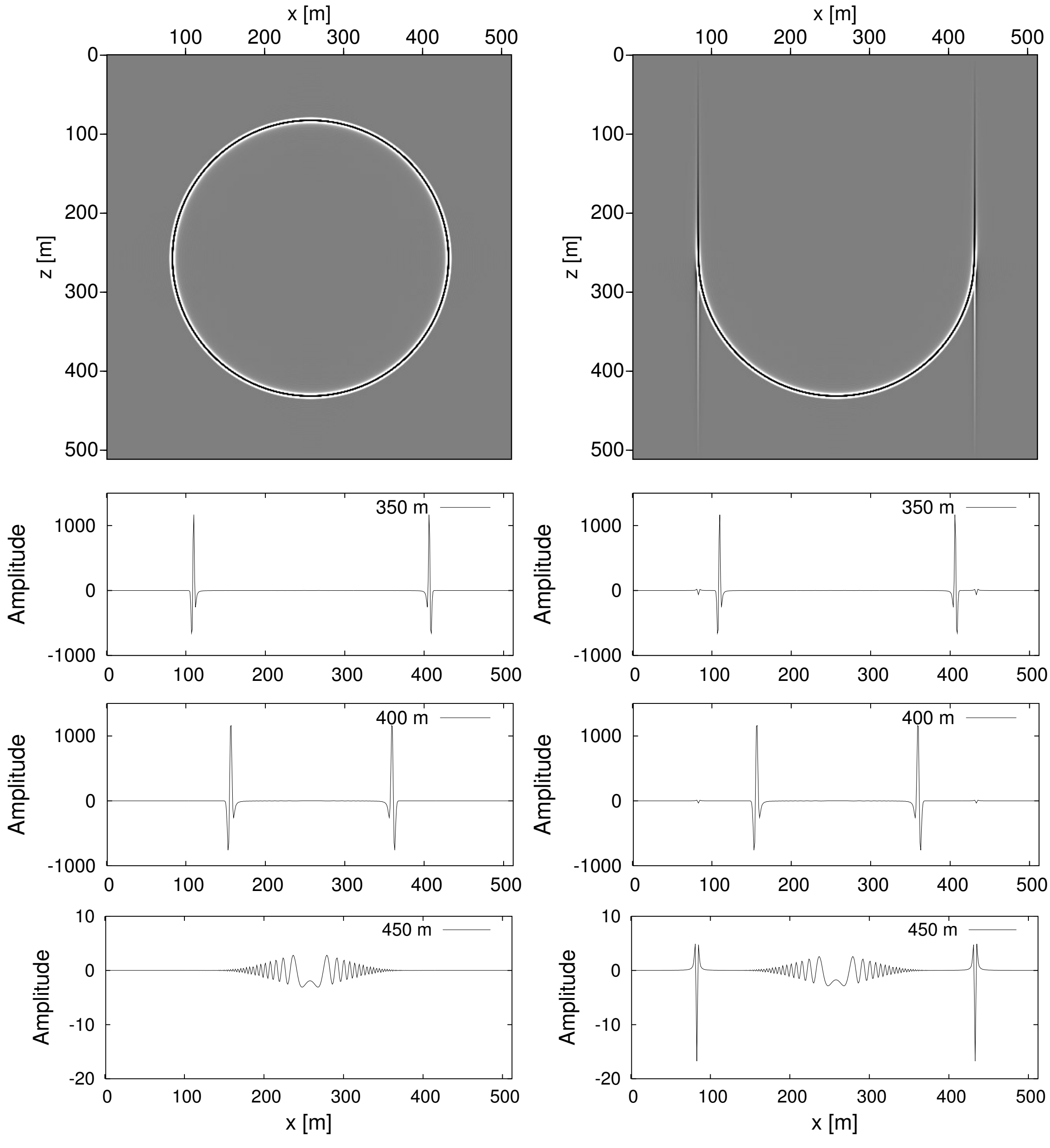

Figure 1: Wave fronts of the two-way and the modified corrected one-way wave equations. Horizontal sections through the wave field at 350 m, 400 m and 450 m depth, respectively are shown at the bottom. Please note the amplitude scale.

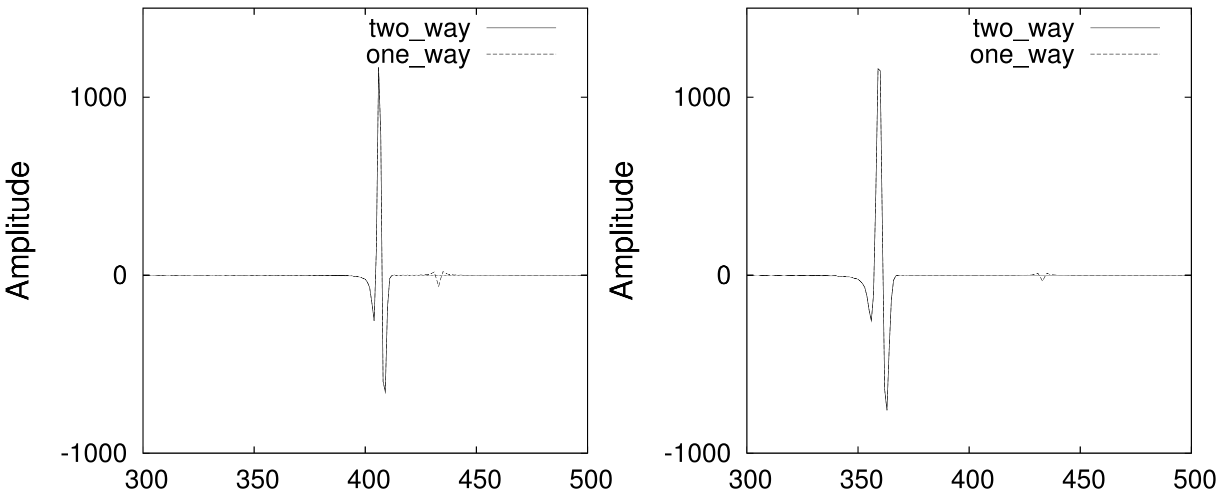

Figure 2: Horizontal sections through the wave field at 350 m (left) and 400 m (right) depth. One-way and two-way wave fields are overlain for comparison.

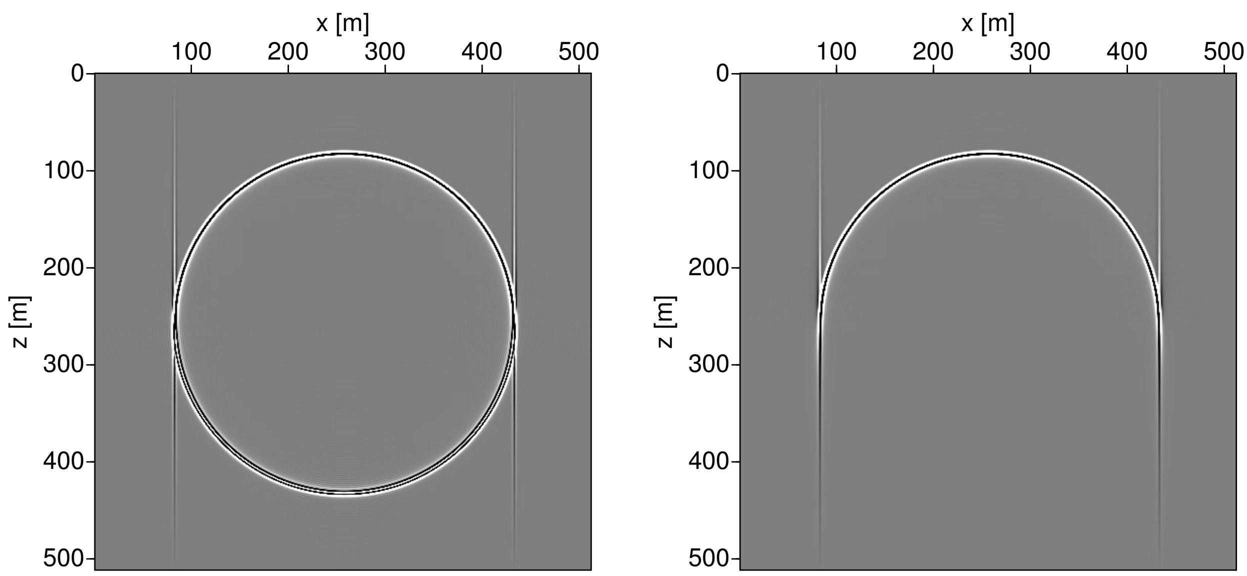

Figure 3: Differences between wave fronts from two-way and one-way wave equations. Left: use of incorrect one-way wave equation does not completely cancel the lower semi-circle of the wave front due to amplitude errors. Right: the lower semi-circle of the wave front is completely canceled. Vanishing of the lower semi-circle of the wave front shows that amplitudes are correct.

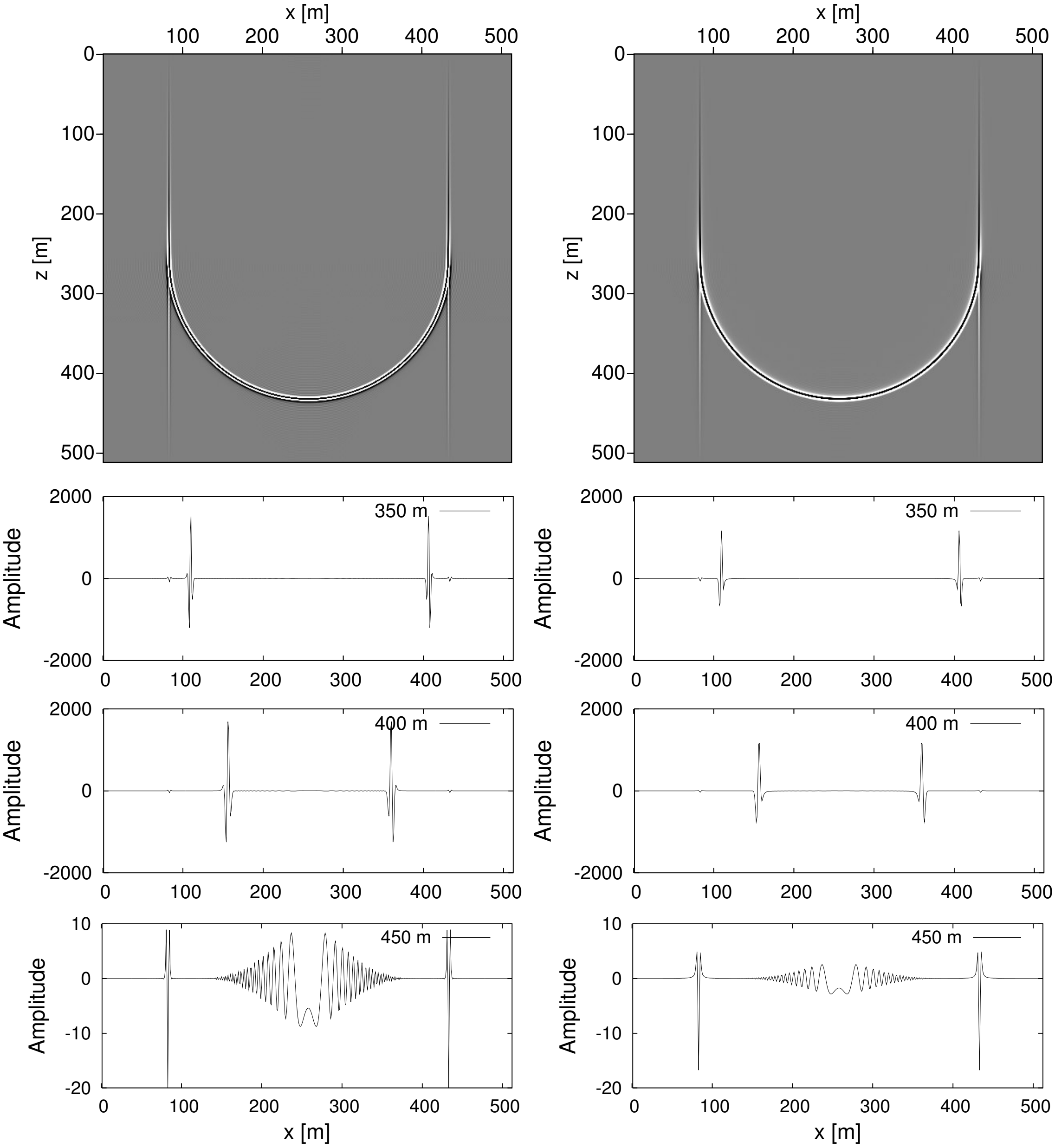

Figure 4: Wave fronts of the one-way wave equation after Baysal et al. (1983) (left) and the corrected one-way wave equation (right). Horizontal sections through the wave field at 350 m, 400 m and 450 m depth, respectively are shown at the bottom. Please note the amplitude scale. Amplitudes in the left panel are about 50% larger than in the right panel.

APPENDIX A

TWO-WAY MODELING BY FOURIER METHOD

We study the simplest problem for wave propagation in homogeneous 2D space. The wave equation reads:

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = \delta(x_1, x_2)\delta(t). \tag{1}$$

We perform the 2D Fourier transform with respect to the x_1 and x_2 coordinates :

$$u(x) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik_1x_1}e^{-ik_2x_2}U(k_1, k_2, t)dk_1dk_2. \tag{2}$$

Then for $U(k_1, k_2, t)$ we obtain the differential equation:

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} + k^2 U = \delta(t), \tag{3}$$

where $k = \sqrt{k_1^2 + k_2^2}$.

Let us solve the equation (3) with help of the temporal Fourier transform: The formal solution of Eq. 3 is given by:

$$u(t) = \pm v^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\frac{1}{\omega^2 - k^2v^2}d\omega. \tag{4}$$

The path of integration in (4) contains zeroes in the denominator. To obtain the exact result of integration we need to define the exact path of integration.

We define the path of integration as follows: in the lower complex ω-halfplane it leads from $-\infty - i\epsilon$ to $\infty - i\epsilon$, where $\epsilon > 0$. Then for $t < 0$ we can close the contour of integration in the lower halfplane ($\Im(\omega) < 0$) and the integral vanishes because there is no singularity inside the contour.

For $t > 0$ we close the contour of integration in the upper halfplane ($\Im(\omega) > 0$) and calculate the integral using the theorem of residues.

This leads to the following result:

$$U = v\frac{\sin(kvt)}{k}, \qquad t > 0,$$
$$U = 0, \qquad t < 0.$$

We obtain the representation of the solution in (x_1, x_2)-space by calculating the inverse Fourier transform (2):

$$u(x_1, x_2, t) = \frac{v}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik_1x_1}e^{-ik_2x_2}\frac{\sin\sqrt{k_1^2 + k_2^2}vt}{\sqrt{k_1^2 + k_2^2}}dk_1dk_2. \tag{5}$$

This leads to the classical result for the 2D case:

$$u(x_1, x_2, t) = \begin{cases} \frac{v}{2\pi}\sqrt{v^2t^2 - x_1^2 - x_2^2}^{\,-1} & \text{for } \sqrt{x_1^2 + x_2^2} < vt \\ 0 & \text{for } \sqrt{x_1^2 + x_2^2} > vt \end{cases}. \tag{6}$$

ONE-WAY WAVE EQUATION FOR THE 2D CASE

Let us now to take the contour of integration in (4) from the second quadrant ($\Re(\omega) < 0, \Im(\omega) > 0$) to the fourth quadrant ($\Re(\omega) > 0, \Im(\omega) < 0$) so that the contour is above the real axis between $\omega = -kv$ to $\omega < 0$ and below the real axis between $\omega > 0$ and $\omega = kv$. For $t > 0$ we can close the contour of integration in the upper halfplane ($\Im(\omega) > 0$) and for $t < 0$ in the lower halfplane ($\Im(\omega) < 0$). Then we have the following solution of equation (3) :

$$U(t > 0) = -i\frac{v}{2k}e^{ikvt}, \tag{7}$$
$$U(t < 0) = -i\frac{v}{2k}e^{-ikvt}. \tag{8}$$

One can test that for $t > 0$

$$\left(\frac{1}{v}\frac{\partial U}{\partial t} - ikU\right) = 0, \tag{9}$$

and for $t < 0$

$$\left(\frac{1}{v}\frac{\partial U}{\partial t} + ikU\right) = 0. \tag{10}$$

We can split a second order equation (3) into 2 first order equations (9) and (10) for $t > 0$ and for $t < 0$. But for $t > 0$ solution (9) does not satisfy the condition

$$U(-k_1, -k_2, t) = U^*(k_1, k_2, t)$$

which guaranties the wave field to be real valued after inverse Fourier transform (2) . Therefore we will use a solution of the form

$$U(k_1, k_2, t) = \frac{v}{2ik\,\mathrm{sign}(k_2)}e^{ivtk\,\mathrm{sign}(k_2)}, \tag{11}$$

where $k = \sqrt{k_1^2 + k_2^2}$. This solution also satisfies the differential equation (9) for $t > 0$.

The wave field in (x_1, x_2)-space can be calculated using the inverse Fourier transform (2)

$$u(x_1, x_2, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{-ik_1x_1}dk_1 \int_{-\infty}^{\infty} e^{-ik_2x_2}U(k_1, k_2, t)dk_2. \tag{12}$$

The result can be written in the following form

$$u(x_1, x_2, t) = \frac{v}{2\pi^2}\int_0^{\infty} \cos k_1x_1 \Re\{A_+\}dk_1, \tag{13}$$

where

$$A_+(k_1, t, x_2) = \int_0^{\infty} e^{-ik_2x_2}\frac{1}{i\sqrt{k_1^2 + k_2^2}}e^{ivt\sqrt{k_1^2+k_2^2}}dk_2. \tag{14}$$

The value of A_+ of (14) can be calculated approximately by the stationary phase method for values $vt >> 1, x_2 = Cvt >> 1$.

$$A_+(k_1, t, x_2) = \int_0^{\infty} e^{ivt\varphi(k_2)}\frac{1}{i\sqrt{k_1^2 + k_2^2}}dk_2, \tag{15}$$

where the phase function

$$\varphi(k_2) = \sqrt{k_1^2 + k_2^2} - k_2C \tag{16}$$

is introduced.

We will now take into account the contribution of the stationary point $k_2^{(0)}$ of phase function (16) and of the end point of integration $k_2 = 0$ to the integral (15) (see Appendix B).

Contribution by the stationary point

The first and second derivatives of the phase function (16) are :

$$\varphi'(k_2) = \frac{k_2}{\sqrt{k_1^2 + k_2^2}} - C,$$

$$\varphi''(k_2) = \frac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}}.$$

The equation for the stationary point

$$\varphi'(k_2) = 0$$

shows that the stationary point exists only for $x_2 > 0$ and the stationary point is at

$$k_2^{(0)} = \frac{Ck_1}{\sqrt{C^2 - 1}} = k_1 \frac{x_2}{\sqrt{v^2t^2 - x_2^2}}.$$

The above required expressions (see Appendix B)

$$\varphi(k_2^{(0)}) = k_1\sqrt{1 - C^2} = \frac{\sqrt{v^2t^2 - x_2^2}}{vt}\, k_1,$$

$$\varphi''(k_2^{(0)}) = \frac{1}{k_1}(1 - C^2)^{\frac{3}{2}} > 0$$

lead to the result

$$A_+(k_1, t, x_2) \approx \sqrt{\frac{2\pi}{vt}} e^{-i\frac{\pi}{4}} (1 - C^2)^{-\frac{1}{4}} \frac{1}{\sqrt{k_1}} e^{ik_1\sqrt{v^2t^2 - x_2^2}}. \tag{17}$$

By introducing the following notations in (17):

$$M = \sqrt{\frac{2\pi}{vt}}(1 - C^2)^{-\frac{1}{4}},$$

$$\alpha = \sqrt{v^2t^2 - x_2^2},$$

the contribution to the wave field from the stationary point is given by

$$u(x_1, x_2, t) = \frac{v}{2}\frac{1}{\pi^2} M \frac{\sqrt{2}}{2} \int_0^\infty \cos(k_1x_1)\Big(\cos(k_1\alpha) + \sin(k_1\alpha)\Big)\frac{1}{\sqrt{k_1}} dk_1.$$

For the area behind of the wave front:

$$0 < x_1 < \alpha = \sqrt{v^2t^2 - x_2^2}$$

using the Fresnel integral (see Appendix B) we have

$$u \approx \frac{v}{2}\frac{1}{\pi}\frac{1}{\sqrt{v^2t^2 - r^2}}.$$

If $x_1 > \alpha = \sqrt{v^2t^2 - x_2^2}$ (in front of the wave front)

$$u \approx \frac{1}{x_1}.$$

For the halfspace $x_2 > 0$ the contribution of the stationary point reproduces behind the wave front the value of the wave field which coincides asymptotically with exact solution (6) of the 2D two-wave equation solution.

Contribution from the end point of integration

We need the following function values (see Appendix B):

$$\varphi(k_2 = 0) = k_1,$$

$$\varphi'(k_2 = 0) = -C,$$

$$A_+(k_1, t, x_2) \approx -\frac{1}{x_2}\frac{1}{k_1}e^{ivtk_1}.$$

For the wave field we get:

$$u(x_1, x_2, t) \approx -\frac{v}{2\pi^2}\frac{1}{x_2}\left\{\frac{1}{2}\int_0^\infty \frac{1}{k_2}\cos\left[k_1(x_1 - vt)\right] + \frac{1}{2}\int_0^\infty \frac{1}{k_2}\cos\left[k_1(x_1 + vt)\right]\right\}.$$

So the contribution of the end point of integration produces two waves with wave fronts orthogonal to the axis $x_2 = 0$ which propagate in opposite directions along the x_1-axis.

2D POINT SOURCE APPROXIMATION

For the implementation of the described 2D one-way equation (11) we need to modify the point source space distribution because equation (11) contains $k = 0$ in the denominator.

We start with the point source distribution in a δ-function-like form

$$p(x_1, x_2; a) = \frac{a}{\pi}e^{-a(x_1^2+x_2^2)} \tag{18}$$

with the normalization

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, x_2; a)dx_1dx_2 = 1.$$

If $a \to \infty$ then

$$p(x_1, x_2; a) \to \delta(x_1, x_2).$$

The space spectrum of the source (18) is given by:

$$P(k_1, k_2; a) = e^{-\frac{(k_1^2+k_2^2)}{2a}} = \frac{a}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2)a}e^{-ik_1x_1}e^{-ik_2x_2}dx_1dx_2. \tag{19}$$

The above source distribution will be introduced by calculating the derivative with respect to a on both sides of the equation (19) :

$$(k_1^2 + k_2^2)\frac{1}{2a^2}e^{-\frac{(k_1^2+k_2^2)}{2a}} =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{1 - a(x_1^2 + x_2^2)\}e^{-(x_1^2+x_2^2)a}e^{-ik_1x_1}e^{-ik_2x_2}dx_1dx_2. \tag{20}$$

Since the source term (20) has the factor k^2 in k space implementation of the one-way 2D wave equation (11) can be done. Division by k is no problem any longer for this kind of source representation.

APPENDIX B

The approximate value of the integral

$$\int_a^b g(t)e^{i\lambda f(t)}dt$$

can be calculated for values $\lambda >> 1$ by the following formula (e.g. Bleistein (1984)) :

$$\int_a^b g(t)e^{i\lambda f(t)}dt \sim \sqrt{\frac{2\pi}{\lambda}} \frac{1}{\sqrt{|f''(t_0)|}} g(t_0)e^{i\lambda f(t_0)}e^{i\frac{\pi}{4}\,\mathrm{sign}(f''(t_0))} + \frac{1}{i\lambda}\frac{g(b)}{f'(b)}e^{i\lambda f(b)} - \frac{1}{i\lambda}\frac{g(a)}{f'(a)}e^{i\lambda f(a)}.$$

The definite Fresnel integrals are (e.g. Gradsteyn and Ryzhin (1965)):

$$\int_0^\infty \sin(t^2)dt = \int_0^\infty \cos(t^2)dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

The definite integral is:

$$\int_0^\infty e^{-at^2}\cos(2xt)dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{x^2}{a}}, \qquad a > 0.$$