IMPEDANCE-TYPE APPROXIMATIONS OF THE P–P AND P–S REFLECTION COEFFICIENTS AND PREDICTION OF ELASTIC PARAMETERS

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keywords: *P–P and P–S reflection coefficient, elastic parameters, impedance function*

ABSTRACT

In the last two decades, many approximations for the P–P reflection coefficient have been proposed. Most of them are derived from the classical Aki and Richards' weak-contrast approximation, using additional assumptions. More recently, it was introduced a new kind of approximation using the concept of an "angular" impedance function. In this work we discuss the impedance-type approximations for the P–P reflection coefficient and we present a corresponding shear reflection impedance for the P–S reflection coefficient. Based on this kind of approximation, we also introduce new indicators for predicting some of the elastic parameters.

INTRODUCTION

Approximations to the Zoeppritz equations, based on Taylor series, can be derived with some assumptions in the model, like restrictions on the elastic-parameter contrasts, or at the incidence angle. The most classical, and used, one is the weak-contrast approximation of Aki and Richards (2002). Recently, some authors (Connolly, 1999; Santos and Tygel, 2004) have shown that the approximations for the P–P reflection coefficient (R_{PP}) using the *Impedance Function* concept provide good results. The idea of this kind of approximation is to look for a representation similar to the expression for the elastic reflection coefficient in the case of normal incidence. In addition, Duffaut et al. (2000) have proposed an impedance type approximation for the P–S reflection coefficient (R_{PS}), based on the work of Connolly (1999).

In Whitcombe et al. (2002) was presented how to use the elastic impedance function to predict some elastic parameters such as Lamé parameters or bulk modulus. They used the three term approximation of the reflection coefficient R_{PP} from Shuey (1985), to obtain the reflectivities of such parameters. They had to suppose a velocity ratio constant and a Gardner relation between compressional velocity and density.

In Grosfeld and Santos (2005) was introduced a new indicator to separate shale over gas sand from shale over brine sand from clastic sections. This indicator is based on the impedance type approximation for the reflection coefficient. In addition, they showed how to use this indicator to estimate a ratio of Lamé parameter λ directly from the reflection coefficient R_{PP}, following the work of Whitcombe et al. (2002) but tryng to be a little more general.

In this work we review the impedance type approximation for R_{PP} and introduce a new impedance type approximation for R_{PS}, following the same ideas proposed by Santos and Tygel (2004). In addition, we show how to estimate density ratio from impedance type approximation for R_{PS}. We also propose an experimental estimation for the Poisson ratio.

IMPEDANCE-TYPE APPROXIMATIONS FOR R_{PP}

Different approximations based on Taylor series for the reflection coefficient R_{PP} exist in the literature. The most famous one is the weak-contrast approximation of Aki and Richards (2002),

$$R_{PP} \approx \frac{1}{2}\left[1 - 4\frac{\beta^2}{\alpha^2}\sin^2\theta\right]\frac{\Delta\rho}{\rho} + \frac{\sec^2\theta}{2}\frac{\Delta\alpha}{\alpha} - 4\frac{\beta^2}{\alpha^2}\sin^2\theta\frac{\Delta\beta}{\beta}, \tag{1}$$

where θ denotes the incidence/transmission angle, α denotes the compressional velocity, β denotes the shear velocity, and ρ denotes the density. We also use the notation $u = (u_2 + u_1)/2$ and $\Delta u = u_2 - u_1$ for $u = \theta, \alpha, \beta, \rho$, where the sub-indices 1 and 2 refer to the incidence and transmission sides of the interface, respectively.

Shuey (1985) rewrote equation (1) in the form,

$$R_{PP} \approx A + B\sin^2\theta + C[\tan^2\theta - \sin^2\theta], \tag{2}$$

where the parameters A (Intercept), B (Gradient) and C are given by

$$A = \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\right], \quad B = \frac{1}{2}\frac{\Delta\alpha}{\alpha} - 2\frac{\beta^2}{\alpha^2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right] \quad \text{and} \quad C = \frac{1}{2}\frac{\Delta\alpha}{\alpha}. \tag{3}$$

For $\theta \leq 30^\circ$, $\tan^2\theta \approx \sin^2\theta$, and then equation (2) can be approximated by,

$$R_{PP} \approx A + B\ \sin^2\theta, \tag{4}$$

which is the most popular AVO formula (from Amplitude Variation with Offset.)

Another kind of approximation can be obtained by following the simple case of normal incidence in elastic media and/or for general oblique incidence in acoustic media. The idea is to approximate the reflection coefficient with the help of an "angular" impedance function $I_j = I(\rho_j, \alpha_j, \beta_j, \theta_j)$, $j = 1, 2$. Such approximation is given by

$$R_{PP} \approx \frac{I_2 - I_1}{I_2 + I_1} = \frac{1}{2}\frac{\Delta I}{I}, \tag{5}$$

where $\Delta I = I_2 - I_1$ and $I = (I_2 + I_1)/2$.

Connolly (1999) with the assumptions of a constant ratio $k = \beta_1/\alpha_1 = \beta_2/\alpha_2$ and a constant angle $\theta = \theta_1 = \theta_2$, introduced the *Elastic* impedance function,

$$I_j = EI_j = N_0\ \alpha_j^{\sec^2\theta}\ \beta_j^{-8k^2\sin^2\theta}\ \rho_j^{1-4k^2\sin^2\theta}, \quad j = 1, 2, \tag{6}$$

where N_0 is a normalization constant (Whitcombe, 2002).

Santos and Tygel (2004) have shown that no exact closed-form solution for equation (5) exists. However, under suitable restrictions in the medium parameters they introduce the *Reflection* impedance function,

$$I_j = RI_j = M_0\ \frac{\rho_j\ \alpha_j}{\sqrt{1 - \alpha_j^2 p^2}}\exp\{-4p^2[\beta_j^2 + f(\beta_j)\}, \tag{7}$$

where M_0 is a normalization constant, p is the ray parameter,

$$p = \frac{\sin\theta_1}{\alpha_1} = \frac{\sin\theta_2}{\alpha_2}, \tag{8}$$

and f is a function that relates ρ with β. For the particular choice of a Gardner's type relationship, $\rho = b\ \beta^\gamma$, where b and γ are constants, f is given by $f(\beta) = \gamma\beta^2/2$. In the derivation of the reflection impedance function RI, it is p, and not θ, that is considered the same on both sides of the interface, honoring Snell's law given by equation (8).

Figure 1 compares the two impedance-type approximations, equations (5)–(7), with the weak-contrast approximation, equation (1), for an interface of shale over gas sand, taken from Castagna and Smith (1994). On the left it is presented a small contrast model ($\Delta\alpha/\alpha = -0.0088, \Delta\beta/\beta = 0.0056, \Delta\rho/\rho = -0.0119$), where we can observe that up to 30^o all the approximations have the same behaviour. Above that, the elastic-impedance-type approximation fails. On the right, a large contrast model ($\Delta\alpha/\alpha = 0.3454,\ \Delta\beta/\beta = 0.4798,\ \Delta\rho/\rho = 0.0207$) is shown. In this case, only the reflection-impedance-type approximation follows the exact curve in the critical region ($\theta \geq 40^o$).

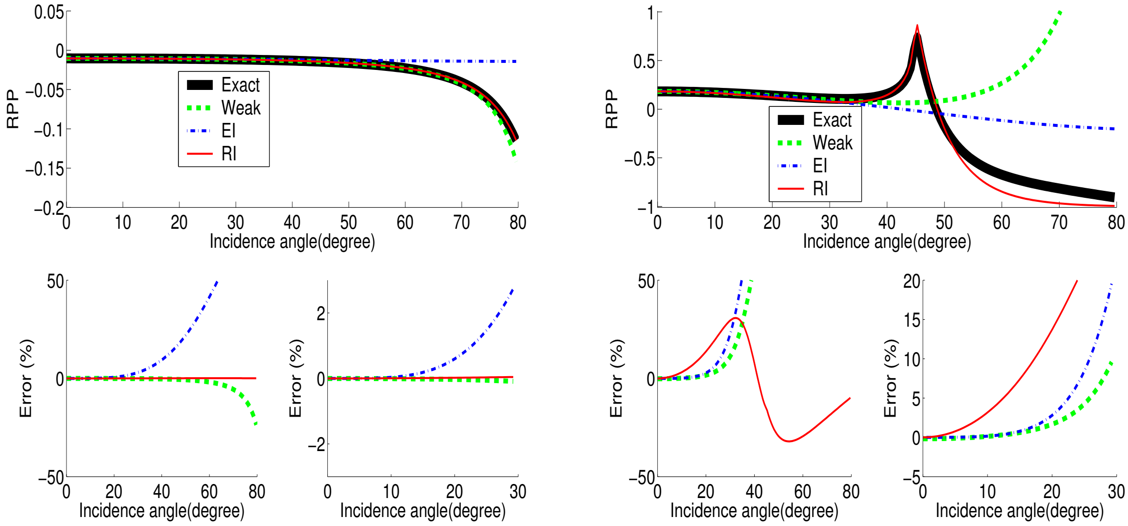

Figure 1: Comparison between the exact P–P reflection coefficient curve and the weak-contrast (weak), elastic-impedance-type (EI) and reflection-impedance-type (RI) approximations, for a small (left) and a large (right) contrast model.

IMPEDANCE-TYPE APPROXIMATIONS FOR R_{PS}

For the P–S reflection coefficient there is not a particular case to analyze, as the case of normal incidence for the P–P reflection. Nevertheless, it is still possible to obtain an impedance-type approximation for the P–S reflection using expression (5). Assuming a weak contrast on the elastic parameters and a small incidence angle, Duffaut et al. (2000) have developed the shear-elastic-impedance (SEI) function, considering the same assumptions as in the previous case of the elastic-impedance for R_{PP}, i.e., $k = \beta_1/\alpha_1 = \beta_2/\alpha_2$ and $\theta = \theta_1 = \theta_2$. The SEI function is given by,

$$I_j = SEI_j = \rho_j^{-m}\, \beta_j^{-n}, \quad j = 1, 2, \tag{9}$$

where,

$$m = [1 + 2k - k\,(1 + 1.5k)\, \sin^2\theta]\, \sin\theta, \quad \text{and} \quad n = 2k\,[2 - (1 + 2k)\, \sin^2\theta]. \tag{10}$$

Following the work of Santos and Tygel (2004), we propose here a shear-reflection impedance. First we define the reflectivity function,

$$I\!\!R = \lim_{\Delta\nu \to 0} \frac{R_{PS}(\nu, \Delta\nu)}{\Delta\nu}, \tag{11}$$

where ν is a single variable describing the change of the elastic parameters, α, β and ρ, and also the angle θ, along the ray and through the interface. For example, $\alpha_1 = \alpha(\nu)$ and $\alpha_2 = \alpha(\nu + \Delta\nu)$. Taking into account the impedance-type approximation given by equation (5), we have

$$I\!\!R = \frac{1}{2}\frac{I'(\nu)}{I(\nu)}, \tag{12}$$

where the prime denotes the derivative with respect to ν.

From the exact expression for the P–S reflection coefficient, it is possible to compute the reflectivity function (11), and then the following differential equation is obtained,

$$\frac{I'}{I} = \frac{-p\,\alpha}{\sqrt{1-\beta^2p^2}}\left[\left(1 - 2\beta^2p^2 + 2\frac{\beta}{\alpha}\sqrt{1-\alpha^2p^2}\sqrt{1-\beta^2p^2}\right)\frac{\rho'}{\rho} \right.$$
$$\left. +4\left(-\beta^2p^2 + \frac{\beta}{\alpha}\sqrt{1-\alpha^2p^2}\sqrt{1-\beta^2p^2}\right)\frac{\beta'}{\beta}\right]. \tag{13}$$

Using Snell's law,

$$p = \frac{\sin\theta}{\alpha} = \frac{\sin\phi}{\beta}, \tag{14}$$

where ϕ is the P–S reflection angle, we can rewite equation (13) as

$$\frac{I'}{I} = -\frac{\sin\theta}{\cos\phi}\left[2\frac{\beta}{\alpha}\cos(\theta+\phi)\left(\frac{\rho'}{\rho}+2\frac{\beta'}{\beta}\right)+\frac{\rho'}{\rho}\right], \tag{15}$$

Approximating the first derivatives of the elastic parameters by the respective finite-difference approximations, i.e., $g' \approx \Delta g/\Delta\nu$, from equation (12) we obtain the first-order approximation for R_{PS},

$$R_{PS} \approx I\!R\ \Delta\nu = -\frac{\sin\theta}{\cos\phi}\left[\frac{\beta}{\alpha}\cos(\theta+\phi)\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta\beta}{\beta}\right)+\frac{1}{2}\frac{\Delta\rho}{\rho}\right], \tag{16}$$

The above equation is the well-known weak-contrast approximation of Aki and Richards (2002) for the P–S reflection coefficient.

Let us now analyze the differential equation (15). From the chain rule,

$$I' = \frac{\partial I}{\partial\rho}\rho' + \frac{\partial I}{\partial\alpha}\alpha' + \frac{\partial I}{\partial\beta}\beta', \tag{17}$$

we conclude that if there was a solution $I = I(\rho(\nu), \alpha(\nu), \beta(\nu), p)$, we would have that function I does not depend explicitly on α, which is not the case. Therefore, some additional assumption, relating α with β and/or ρ, must be done in order to solve (13). In this work we are going to assume a constant ratio $k = \beta/\alpha$. With this assumption only, it is still not possible to find a solution. So, similarly to the case of the P–P reflection, we include a Gardner's-type relation between ρ and β, $\rho = b\ \beta^\gamma$. After some tedious mathematical manipulations we obtain a solution for equation (13), which we call *Shear-Reflection-Impedance* (SRI) function,

$$I_j = SRI_j = \exp\{2\ \phi_j/\ k - (2+\gamma)\,[k\ \theta_j + \sin(\theta_j+\phi_j)]\}, \quad j = 1, 2, \tag{18}$$

where,

$$p = \frac{\sin\theta_1}{\alpha_1} = \frac{\sin\phi_1}{\beta_1} = \frac{\sin\theta_2}{\alpha_2} = \frac{\sin\phi_2}{\beta_2}. \tag{19}$$

Figure 2 compares the two impedance-type (SEI and SRI) approximations for R_{PS}, with the weak-contrast approximation (16) for an interface of shale over gas sand (from Castagna and Smith (1994)). On the left are shown the results for the contrasts, $\Delta\alpha/\alpha = -0.1440, \Delta\beta/\beta = -0.2500, \Delta\rho/\rho = -0.0368$. We can observe that all approximations are reasonable for incidence angles less than 50^o, but for large angles only the approximation based on the SRI has a good fit. On the right are the results for a large contrast model, $\Delta\alpha/\alpha = 0.3454, \Delta\beta/\beta = 0.4798, \Delta\rho/\rho = 0.0207$. In this case, none of the approximations was able to follow the exact curve in the critical region. Moreover, our approximation with the SRI has the worst results near zero offset.

PREDICTION OF ELASTIC PARAMETERS

Fundamental rock properties such as the Lamé parameters (λ and μ) or Poisson ratio (σ) are better understood than velocities or impedances and so, it is desirable to extract them from the data. In that direction, Whitcombe et al. (2002) defined an *Extended-Elastic-Impedance* function, from which they can recover λ, μ and the compressibility module κ. The estimatives are computed directly from the reflection coefficient samples at specific angles of incidence. The approach requires additional assumptions on the elastic parameters, to say the ratios β/α and A/C in equation (2) are constant. In the following we present a similar approach (Grosfeld and Santos, 2005), using the previous derived expressions of the impedance-type approximations for the reflection coefficients.

For any choice of the impedance function, we can define function J as

$$J = \frac{I_1}{I_2} \approx \frac{1-R}{1+R}, \tag{20}$$

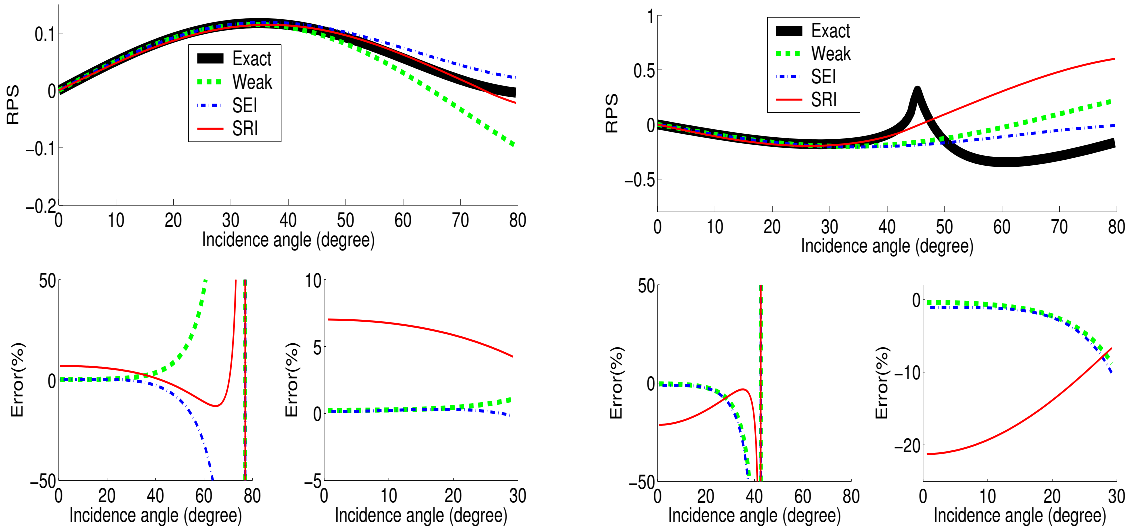

Figure 2: Comparison between the exact P–S reflection coefficient curve and the weak-contrast (weak), shear-elastic-impedance-type (SEI) and shear-reflection-impedance-type (SRI) approximations, for a small (left) and a large (right) contrast model.

where R can be R_{PP} or R_{PS}. Note that this function is directly obtained once you have the reflection coefficient, and it depends on the angle of incidence. We are going to use J to estimate some elastic parameters.

For $\theta = 45^o$ the weak-contrast approximation for R_{PP}, equation (1), gives

$$R_{PP} \approx \left(1 - 2\frac{\beta^2}{\alpha^2}\right)\frac{\Delta\rho}{2\,\rho} + \frac{\Delta\alpha}{\alpha} - 2\frac{\beta^2}{\alpha^2}\frac{\Delta\beta}{\beta} = q\,\frac{\Delta\lambda}{2\,\lambda}, \tag{21}$$

where $q = 1 - 2\,\beta^2/\alpha^2$ and $\lambda = \rho\,(\alpha^2 - 2\,\beta^2)$. From the impedance approximation (5), we can write

$$\frac{\Delta I}{2I} = q\,\frac{\Delta\lambda}{2\lambda}, \tag{22}$$

If we consider the ratio β/α constant along the ray, integration of the above equation gives

$$I(45^o) = C_0\,\lambda^q, \tag{23}$$

where C_0 is a constant. Therefore, from the definition of function J, equation (20),

$$J(45^o) = \left(\frac{\lambda_1}{\lambda_2}\right)^q \equiv r_\lambda^q. \tag{24}$$

The above result suggests a strategy to extract the λ-ratio, r_λ, assuming that additional information about the constant ratio β/α is known. After an AVA procedure is carried out, select the amplitudes of the events related to angles close to 45^o and with the help of equations (20) and (24), estimate r_λ. Equation (21) also indicates that the λ-reflectivity, $I\!R_\lambda = \Delta\lambda/2\lambda$, can be estimate directly from R_{PP}. However, as we are going to show in the next section, this approximation is not a good alternative.

In a similar way it is possible to estimate the ρ-ratio (or the ρ-reflectivity, $I\!R_\rho$) from the P–S reflection coefficient. Let θ_ρ and ϕ_ρ be such that $\theta_\rho + \phi_\rho = 90^o$. From Snell's law (14), we find

$$\theta_\rho = \arctan(\alpha/\beta). \tag{25}$$

Therefore, approximation (16) for R_{PS} at $\theta = \theta_\rho$ reduces to

$$R_{PS} \approx -\frac{\Delta\rho}{2\rho}, \tag{26}$$

and then,

$$\frac{\Delta I}{2I} = -\frac{\Delta \rho}{2\rho}. \tag{27}$$

Therefore, assuming again a constant ratio β/α along the ray, integration of the above equation results in

$$I(\theta_\rho) = \frac{D_0}{\rho}, \quad \text{and} \quad J(\theta_\rho) = \frac{\rho_2}{\rho_1} \equiv \frac{1}{r_\rho}. \tag{28}$$

where D_0 is a constant. Observe that, since $\alpha \geq 2\beta/\sqrt{3}$, the minimal value for θ_ρ is given by $\arctan(2/\sqrt{3}) \approx 49.11^o$. Therefore, θ_ρ belongs to the interval $[50^o, 90^o]$ and ϕ_ρ belongs to $[0^o, 40^o]$.

APPLICATION TO A WELL-LOG DATA

To test the validity of our approximations, equations (24) and (28), we apply the previous formulas to a real well-log data depicted in Figure 3. In all experiments, the value for $q = 1 - 2(\beta/\alpha)^2$ and $\theta_\rho = \arctan(\alpha/\beta)$ were obtained using the average value $\overline{k}$ of the ratios β/α for each sample of the well. We find $\overline{k} = 0.458$, and then $q = 0.585$ and $\theta_\rho \approx 65^o$.

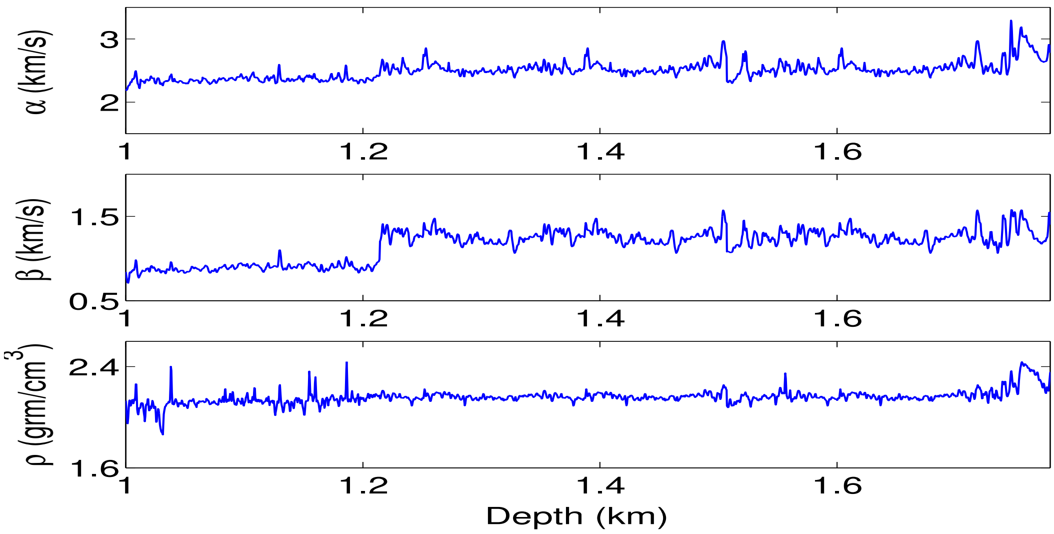

Figure 3: Well-log data: P-velocity (α), S-velocity (β) and density (ρ).

Figure 4 shows the result of applying equation (21) directly, i.e., if we estimate $I\!R_\lambda$ directly from R_{PP}. The relative errors are completely out of an acceptable range. The correlation coefficient between $J(\theta)$ and r_λ^q, plotted on the left of Figure 5 for the whole well, has a maximum near $\theta = 45^o$. From the right side of Figure 5 we may conclude that the approximation of r_λ from $J(45^o)$ is quite good. Both curves, exact and extracted, are almost identical, as we can see from the zoom depicted in the top of Figure 6. To stress the good performance of the reflection impedance function, we have also plotted in the bottom of the same figure the corresponding zoom for the results obtained from the refelction and elastic impedance approximations.

Figure 7 shows the estimatives for $I\!R_\rho$ computed directly from R_{PS}, using equation (26). As in the previous case for R_{PP} the results are unacceptable. In Figure 8 we show the extraction of r_ρ, using equation (28), where we again observe that for $\theta = 65^o$ the correlation coefficient between $J(\theta)$ and $1/r_\rho$ attains its maximum. Moreover, the curves of $J(65^o)$ and $1/r_\rho$ are also very similar. A zoom is shown in Figure 9.

We have also developed an experimental relationship between the Poisson ratio σ, which is considered a good fluid indicator, and some kind of near/far offset relation using J. In Figure 10 we plot the curves of $J(45^o)/J(5^o)$ and $r_\sigma = \sigma_1/\sigma_2$. Observe the good agreement between them along the well (the percentage error is less than 2% almost everywhere). The correlation coefficient between both curves is 0.92. The common behavior of both curves can be better observed in the zoom depicted in Figure 11.

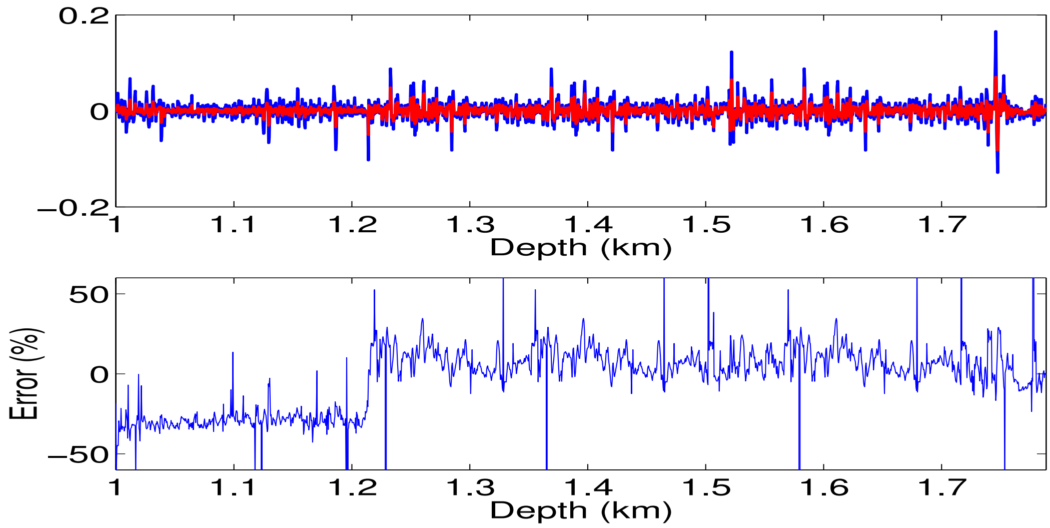

Figure 4: Top: R_λ (dashed line) and $R_{PP}(45^o)/q$ (solid line). Bottom: Percentage error.

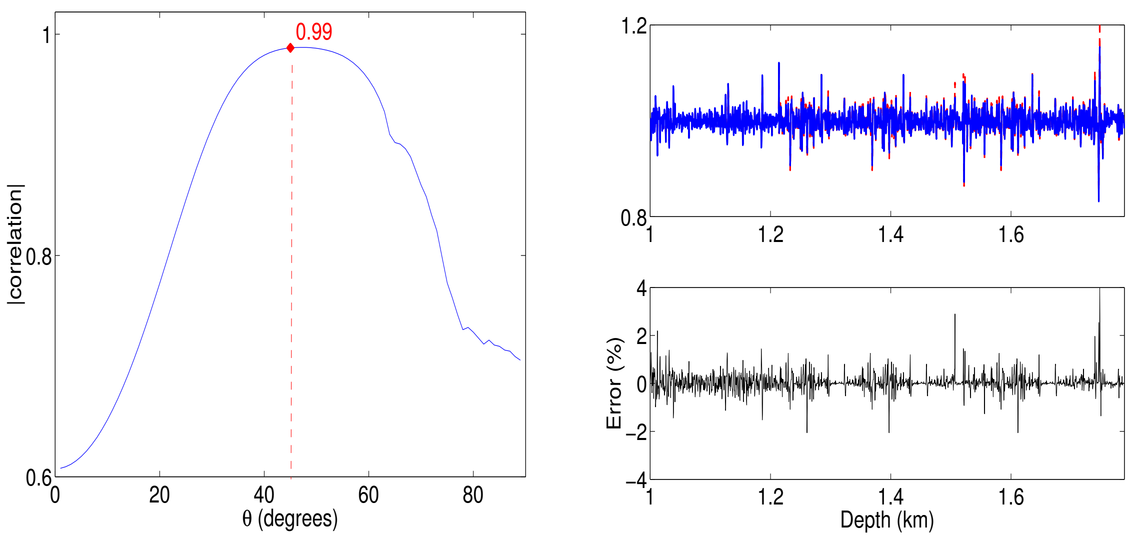

Figure 5: Left: Correlation coefficient between $J(\theta)$ and r^q_λ. Top right: $J(45^o)$ (solid line) and r^q_λ (dashed line). Bottom right: Percentage error.

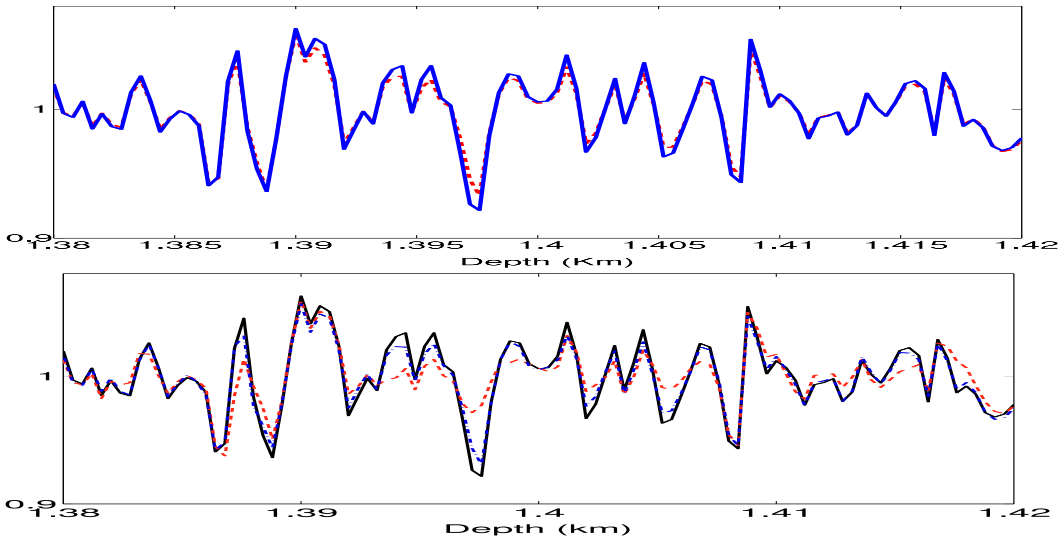

Figure 6: Top: Zoom from the top right picture of Figure 5. Bottom: Comparison between r_λ^q (solid line) and $J(45^o)$ computed from elastic (dashed line) and reflection (dash-dotted line) impedance approximations.

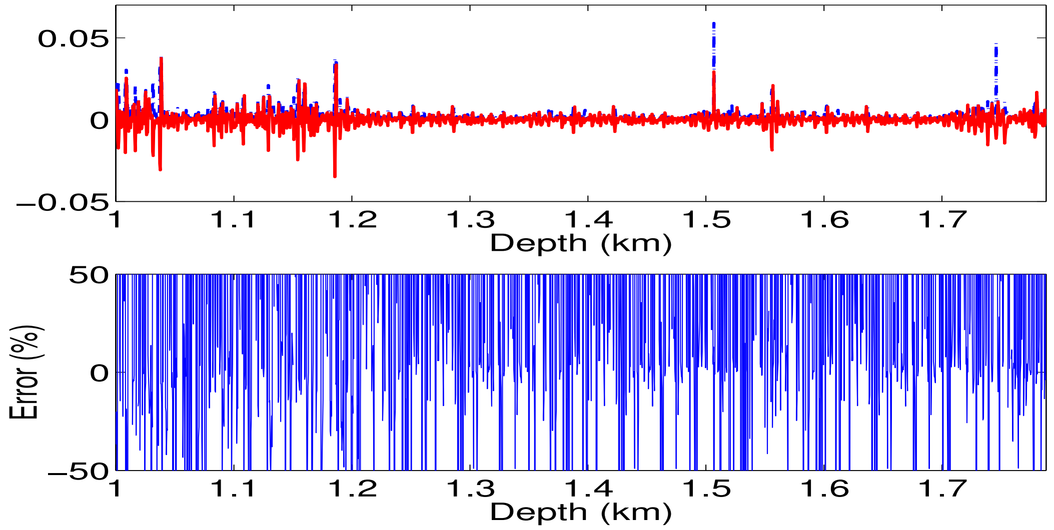

Figure 7: Top: $-I\!\!R_\rho$ (dashed line) and $R_{PS}(65^o)$ (solid line). Bottom: Percentage error.

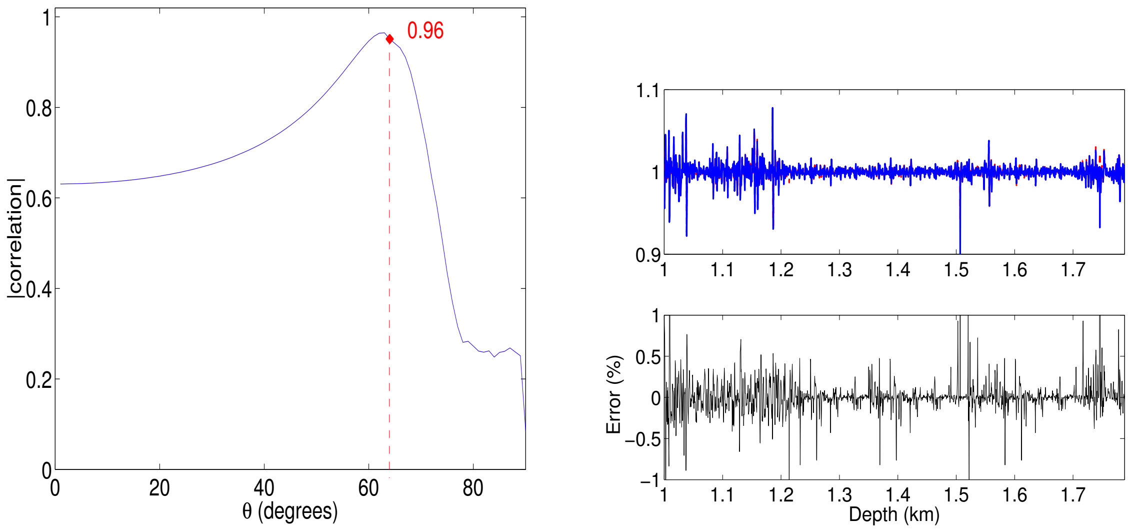

Figure 8: Left: Correlation coefficient between $J(\theta)$ and $1/r_\rho$. Top right: $J(65^o)$ (solid line) and $1/r_\rho$ (dashed line). Bottom right: Percentage error.

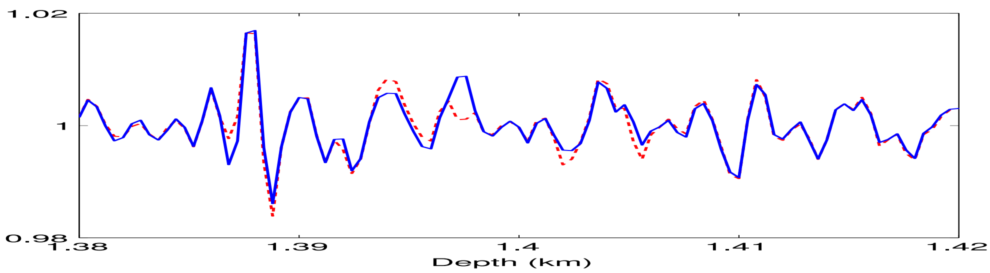

Figure 9: Zoom from the top right of Figure 8 $J(65^o)$ (solid line) and $1/\rho$ (dashed line).

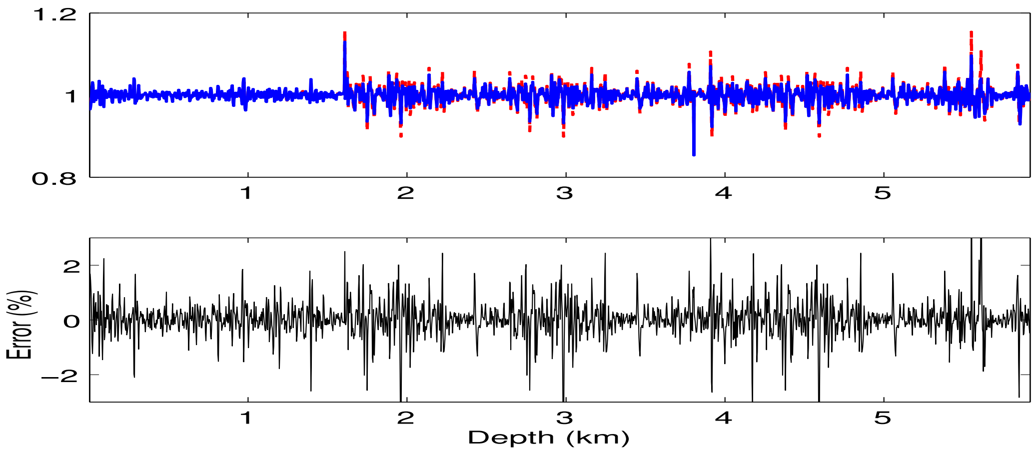

Figure 10: Top: Comparison between the Poisson ratio (dashed line) and $J(45^o)/J(5^o)$ (solid line). Bottom: Percentage error.

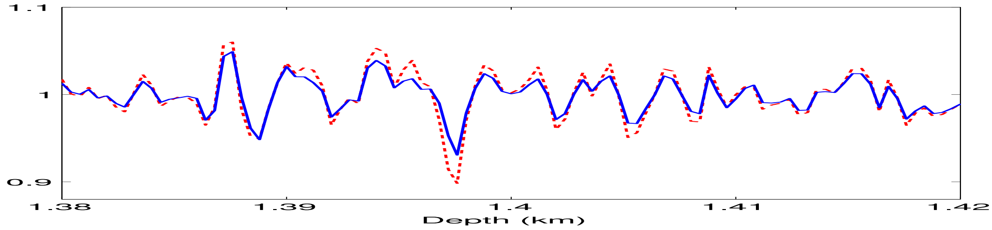

Figure 11: Zoom from the top of Figure 10: Poisson ratio (dashed line) and $J(45^o)/J(5^o)$ (solid line).

CONCLUSIONS

We have presented a brief comparison between weak contrast and impedance type approximations for the P–P reflection coefficient. Our results indicate that the approximation based on the reflection impedance function is better than the others, even in the critical region. In addition, we introduced a reflection-impedance-type approximation for R_{PS}. Unfortunately, the numerical tests have shown that this new approximation does not work properly.

Based on the impedance-type approximation, we have also propose a procedure to obatin the ratios of two elastic parameters, density and Lamé's parameter λ, directly from the reflection coefficients (P–P and P–S). We have applied our approach to a well-log data, with encouraging results.

The next step is to try to estimate other elastic parameters, such as compressibility, or shear rigidity, to obtain a complete set of physical parameters.

ACKNOWLEDGEMENTS

This work has been partially supported by CNPq (307165/2003-5) and FAPESP (02/08544-5 & 03/09839-6), Brazil, and the sponsors of the Wave Inversion Technology (WIT) Consortium, Germany.

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