UNIFORM RESAMPLING USING THE SINC FUNCTION

A.C. Camargo and L.T. Santos

email: *lucio@ime.unicamp.br*
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ABSTRACT

It is common to be needed to reconstruct functions which samples falls on a nonequally spaced grid. This is due to the fact that some of the most used algorithms require samples in a regular (uniform) Cartesian grid. Therefore, it is necessary to make an uniform resampling, *i.e., to interpolate the nonuniform samples in a set of equally spaced points. In this work, it is first shown that the resampling problem can be formulated as a problem of solving a system of linear equations. A solution for this system can be found using the pseudoinverse matrix, a process that is impractical for a large number of variables. From particular characteristics of the resampling problem, it is possible to develop a better algorithm, which only uses a limited number of samples to calculate each uniform sample, transforming the original problem into a sequence of linear systems with less variables. The final result can be viewed as both optimal and computationally efficient. Two applications are presented to demonstrate the efficiency of the new method.*

INTRODUCTION

The problem of handling data that falls on a nonequally spaced grid occurs in numerous fields of science. In general, this problem occurs due to the fact that some very useful algorithms are based on the Discrete Fourier Transform (DFT), which requires that the samples lie over a regular Cartesian grid.

In many seismic-data processing algorithms the input data must be given in an uniform grid. However, for real data, difficulties in positioning the sources and receivers may cause the spatial sampling intervals to vary from place to place. If the variations of the sampling intervals are too large to be acceptable for data processing, seismic traces have to be resampled. Then, we say that is necessary to make an *uniform resampling*.

In this work we show that the uniform resampling problem can be formulated as the solution of a linear system of equations. We also discuss the *Uniform ReSampling* (URS) and the *Block Uniform ReSampling* (BURS) algorithms, developed by Rosenfeld (1998) to solve the problem. The BURS algorithm is both optimal and efficient and is suitable for problems where the number of samples is very large and it is necessary a fast method to reconstruct the function. We apply the algorithm to three data sets to show that the results are of excellent quality.

ONE-DIMENSIONAL CASE

Let us consider a continuous real function f, sampled in a finite set of nonequally spaced points, $\{\tau_1, \tau_2, \ldots, \tau_m\}$, $\tau_i \in I\!R$, $i \in M = \{1, 2, \ldots, m\}$. The uniform resampling problem consists in to find an approximation to the function on an equally spaced set of points, i.e., to approximate $f(t_j)$, $t_j = t_0 + j\Delta t$, $t_0 \in I\!R$, $j \in N = \{1, 2, \ldots, n\}$. Such problem can be solved using the following theorem due to *Claude Shannon* in 1949 (see, e.g., Briggs and Henson (1995)):

SHANNON SAMPLING THEOREM: Let f be a band-limited real function, i.e., its Fourier Transform, $\widehat{f}$, is such that $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega > 0$. If $\Delta t < \pi/\Omega$, then for any $t_0 \in I\!R$

$$f(\tau) = \sum_{n=-\infty}^{\infty} f(t_0 + n\Delta t) \text{ sinc } \left(\frac{\tau - n\Delta t - t_0}{\Delta t} \right), \tag{1}$$

where

$$\text{sinc } t = \frac{\sin \ \pi t}{\pi t}, \tag{2}$$

Equation (2) defines the so called *cardinal sine*, or simply *sinc function*. For more details about this function, see Gearhart and Shultz (1990). Equations (1) and (2) are readily extend to higher dimensions by replacing the sum by a multiple sum and the sinc function by a product of sinc functions.

From equation (1), for each τ_i, $i \in M$ we can write

$$f(\tau_i) \approx \sum_{j=1}^{n} f(t_j) \text{ sinc } \left(\frac{\tau_i - t_j}{\Delta t} \right), \tag{3}$$

which can be viewed as a system of linear equations

$$Ax = b, \tag{4}$$

where the elements of the matrix $A \in I\!R^{m\times n}$, of the vector $x \in I\!R^n$ and of the vector $b \in I\!R^m$ are given by $a_{ij} = \text{sinc}[(\tau_i - t_j)/\Delta t]$, $x_j = f(t_j)$ and $b_i = f(\tau_i)$, for all $i \in M$ and $j \in N$. Thus, our problem is one of solving a set of m linear equations with n unknowns.

UNIFORM RESAMPLING ALGORITHM

As in general the matrix A is not square and, moreover, in most practical cases $m > n$, a solution for this problem can be found using the (Moore-Penrose) pseudoinverse of A,

$$x = A^+ b. \tag{5}$$

The matrix A^+ has n rows and m columns and satisfies the relations $A^+AA^+ = A^+$ and $AA^+A = A$. The pseudoinverse provides the optimal solution to the equation (4) in the minimal-norm least-square sense, i.e., it selects among all vectors x which minimize the expression $\|Ax - b\|$, the one with minimal norm $\|x\|$. Here, $\|\cdot\|$ denotes the Euclidian norm, $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.

The *Uniform ReSampling* (URS) algorithm computes the solution $x_{URS} = A^+ b$ using the *Singular Value Decomposition* (SVD), which is a standard component of most mathematical software packages (see, e.g., Trefethen and Bau (1997)). Although x_{URS} is, in some sense, an optimal solution, it has two inherent drawbacks. First, the computation of A^+ becomes impractical when the dimensions of A are too large. When m e n are on the order of several hundreds, inversion is practical. Second, each uniform sample, say x_j, is calculated by multiplying the jth row of A^+ by the vector b, i.e., m multiplications (and $m - 1$ additions) are involved. Using the fact that measurements that are distant from the point t_j will have coefficients with small magnitude, Rosenfeld (1998) created an algorithm that includes only a limited number of terms in its computation. In the following section, this algorithm is developed.

BLOCK UNIFORM RESAMPLING ALGORITHM

Rosenfeld (1998) developed a new algorithm to find a solution of the form $x = \bar{A}^+ b$, where each row of the matrix $\bar{A}^+$ contain mostly zeroes and only a restricted number of nonzero coefficients, concentrated in the neighborhood of the line corresponding to t_j. The steps of the *Block Uniform ReSampling* (BURS) algorithm is as follows: For each t_j, instead of considering all the m nonuniform points, select only $\bar{m}$ points in the vicinity of t_j; for example, one could include all the points within a radius δ from t_j. Similarly, it is sufficient to estimate only $\bar{n}$ uniform samples of the result. For example, select all grid points within a radius Δ from t_j. Rosenfeld (1998), after many trials and errors, suggests the relation $\Delta \geq 1.5\ \delta$. The following system of equations is obtained

$$B_j x^j = b^j, \tag{6}$$

where B_j is an $\bar{m} \times \bar{n}$ matrix of interpolation coefficients (a submatrix of A), x^j is a $\bar{n}$-dimensional subvector of x, and b^j is a $\bar{m}$-dimensional subvector of b, which contains the participating measurements. The solution of equation (6) is not computed, but the pseudoinverse of B_j using the SVD.

The next step is to isolate the row of B_j^+ that corresponds to t_j. This row contains $\bar{m}$ elements, which are the desired coefficients of the linear combination for which x_j is the best (in the minimal-norm least-squares sense). These values are now inserted into the appropriated positions in the matrix $\bar{A}^+$. That is, the entire j-th row of the matrix $\bar{A}^+$ is set to zero, with the exception of these $\bar{m}$ coefficients, which are placed in the positions corresponding to their respective measurements (b vector). This is done for each uniform coordinate t_j. The result is an $n \times m$ matrix $\bar{A}^+$, which contains mostly zeroes, except for a narrow band along its "diagonal".

The following scheme summarizes the method:

BURS ALGORITHM

- Given $f(\tau_i)$, $\tau_i \in I\!R$, $i \in M = \{1, 2, \ldots, m\}$, $t_j = t_0 + j\Delta t$, $t_0 \in I\!R$, $j \in N = \{1, 2, \ldots, n\}$, $\delta > 0$ and $\Delta \geq 1.5\delta$.
- For $j = 1, \ldots, n$
  - Let $d = \{k \in \mathrm{M} \;\mid\; |\tau_k - t_j| < \delta\}$ and $D = \{l \in \mathrm{N} \;\mid\; |t_l - t_j| < \Delta\}$.
  - Let B_j be the submatrix of A composed by the elements $a_{kl},\ k \in d,\ l \in D$.
  - Compute B_j^+.
  - Isolate the row of B_j^+, which corresponds to t_j, and plug it into the appropriate location in the $j-$th line of $\bar{A}^+$.
- Let $b = (f(\tau_1), \ldots, f(\tau_m))^T$.
- The uniform samples are given by $x_{BURS} = \bar{A}^+ b$, and then, $f(t_j) \approx (x_{BURS})_j$.

The BURS algorithm is very efficient because many elements of the inversion matrix $\bar{A}^+$ are zero (the j-th row of $\bar{A}^+$ contains only $\bar{m}$ nonzero coefficients), and one has only to keep track of these nonzero coefficients.

TWO-DIMENSIONAL CASE

Let us consider a continuous real function $f : I\!R^2 \to I\!R$, sampled in a finite set of nonequally spaced points, $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)\}$. We want to find an approximation for $f(a_i, b_j)$, $(a_i, b_j) = (a_0, b_0) + (i\Delta a, j\Delta b)$, $(a_0, b_0) \in I\!R^2$, $i \in N_1 = \{1, 2, \ldots, n_1\}$ and $j \in N_2 = \{1, 2, \ldots, n_2\}$.

Such problem can be solved applying Shannon's theorem to higher dimensions. For each (α_k, β_k), $k \in M = \{1, 2, \ldots, m\}$, we have

$$f(\alpha_k, \beta_k) \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} f(a_i, b_j) \text{ sinc}\left(\frac{\alpha_k - a_i}{\Delta a}\right) \text{ sinc}\left(\frac{\beta_k - b_j}{\Delta b}\right). \tag{7}$$

We can sort these Cartesian grid points by columns (or rows) in a vector as follows: $\{(\widehat{a}_1, \widehat{b}_1), (\widehat{a}_2, \widehat{b}_2), \ldots, (\widehat{a}_n, \widehat{b}_n)\}$, where $(\widehat{a}_1, \widehat{b}_1) = (a_1, b_1)$, $(\widehat{a}_2, \widehat{b}_2) = (a_1, b_2), \ldots, (\widehat{a}_{n_2},\ \widehat{b}_{n_2}) = (a_1, b_{n_2}),\ (\widehat{a}_{n_2+1}, \widehat{b}_{n_2+1}) = (a_2, b_1), \ldots,\ (\widehat{a}_n, \widehat{b}_n) = (a_{n_1}, b_{n_2})$, where $n = n_1 n_2$. Then, from equation (7),

$$f(\alpha_k, \beta_k) \approx \sum_{\ell=1}^{n} f(a_\ell, b_\ell) \text{ sinc}\left(\frac{\alpha_k - \widehat{a}_\ell}{\Delta a}\right) \text{ sinc}\left(\frac{\beta_k - \widehat{b}_\ell}{\Delta b}\right),\ k \in M. \tag{8}$$

Let us observe that as in the one-dimensional case, the above set of equations form a linear system $AX = B$. The elements of matrix $A \in I\!R^{m \times n}$, of vector $X \in I\!R^n$ and of vector $B \in I\!R^m$ are given by $a_{k\ell} = \text{sinc}[(\alpha_k - \widehat{a}_\ell)/\Delta a]\, \text{sinc}[(\beta_k - \widehat{b}_\ell)/\Delta b]$, $\quad X_\ell = f(\widehat{a}_\ell, \widehat{b}_\ell)$, $\quad B_k = f(\alpha_k, \beta_k)$, $\quad k \in M$,

$\ell \in N = \{1, 2, \ldots, n\}$. The solution for this linear system is given by

$$X = A^+ B, \tag{9}$$

and all the previous methods discussed in one-dimensional case can be applied.

NUMERICAL EXPERIMENTS

As a first example, we apply the described algorithms to the one-dimensional function $f(x) = 15x^5 + e^{x/2}\ \cos x - x^9$ sampled onto $m = 256$ nonuniform points. The image was reconstructed onto $n = 128$ uniform grid using both URS and BURS algorithms. As expected, the BURS algorithm had a better performance, as shown in Figure (1). For the URS algorithm, it was computed the pseudoinverse of an (256×128) matrix while in the BURS algorithm it was computed 128 pseudoinverses with sizes varying from 7×5 to 17×11.

To show the potential of the interpolation scheme for seismic purposes, and as another one-dimensional example, we resampled a seismic section where the receivers are not equally spaced. Both algorithms, URS and BURS were applied to simulate the corresponding seismic section for equally spaced receiver locations. Figures 2–5 show the results, where again we observe a better performance for the BURS algorithm.

Finally, as a two-dimensional example, we apply the BURS algorithm to reconstruct an image with some missing samples. The chosen image was the central part of a Lena image with 256×256 pixels, resulting in an image of 128×128 pixels. We randomly removed 4916 pixels and resampled the image using BURS. In Figure 6 we can see the original Lena image, the corrupted image and the reconstructed one. The running time was 2hs29min and the error is 6.7%. The error was measured by the Frobenius norm of the difference between the original image and the obtained one divided by the Frobenius norm of the original image. Algorithm URS was computationally impractical for this problem: the running time exceeded 5 hours.

CONCLUSIONS

We discussed a new gridding algorithm that is both optimal and efficient. The original problem of resampling over a uniform grid was first formulated as a problem of solving a set of linear equations. The solution is obtained using the pseudoinverse (SVD). This method, the URS algorithm, is optimal in the minimal-norm least-square sense. The BURS algorithm is a suboptimal counterpart of the URS method, which is efficient and practical for large problems and others situations. Only a limited number of measurements are used to generate each uniform grid point. An appropriate set of linear equations is constructed and subsequently solved using the SVD.

Both BURS and URS algorithms were applied in resampling of seismograms for simulated a seismic section with equally spaced receivers. It was shown that BURS algorithm gave better results. In the two-dimensional case, BURS was used to reconstruct an image and the result was very satisfactory.

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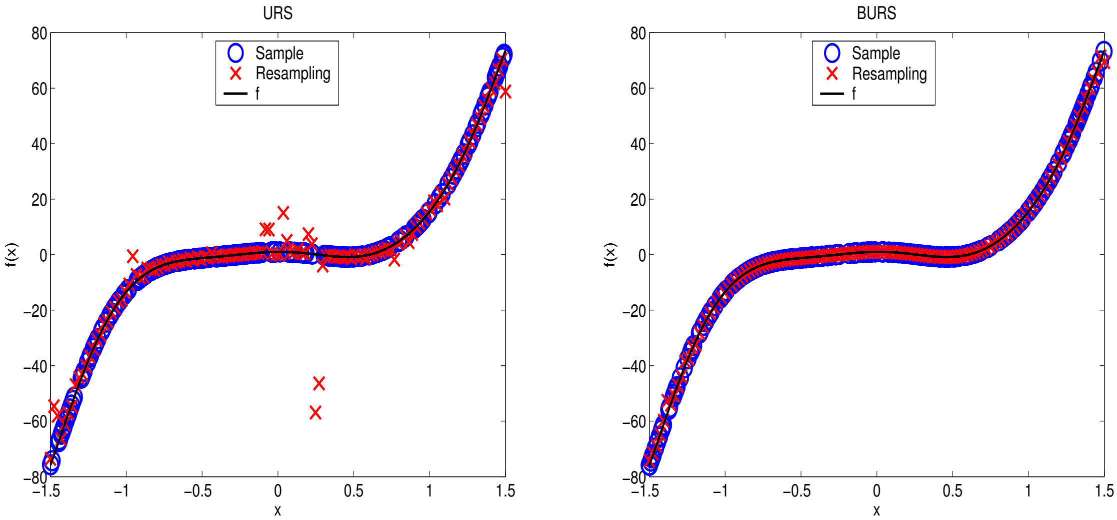

Figure 1: Uniform resampling of function $f(x) = 15x^5 + e^{\frac{x}{2}} \cos x - x^9$ using the URS (left) and BURS (right) algorithms, with $\delta = 0.17$ and $\Delta = 0.30$.

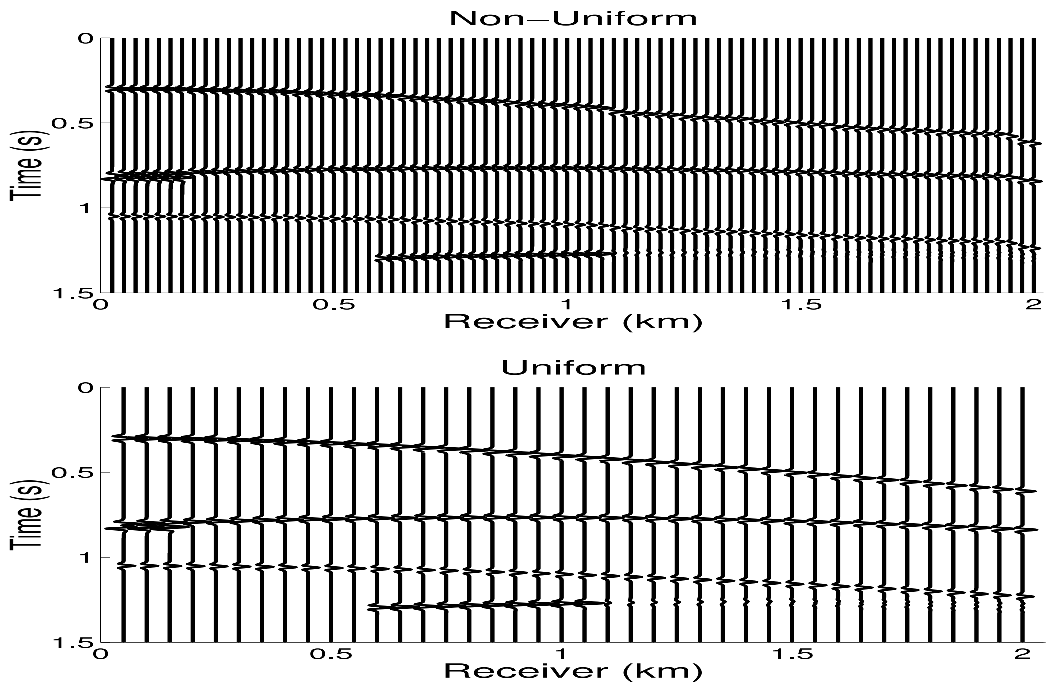

Figure 2: Top: Modeled seismic section for nonequally spaced receiver locations. Bottom: Modeled seismic section for equally spaced receiver locations.

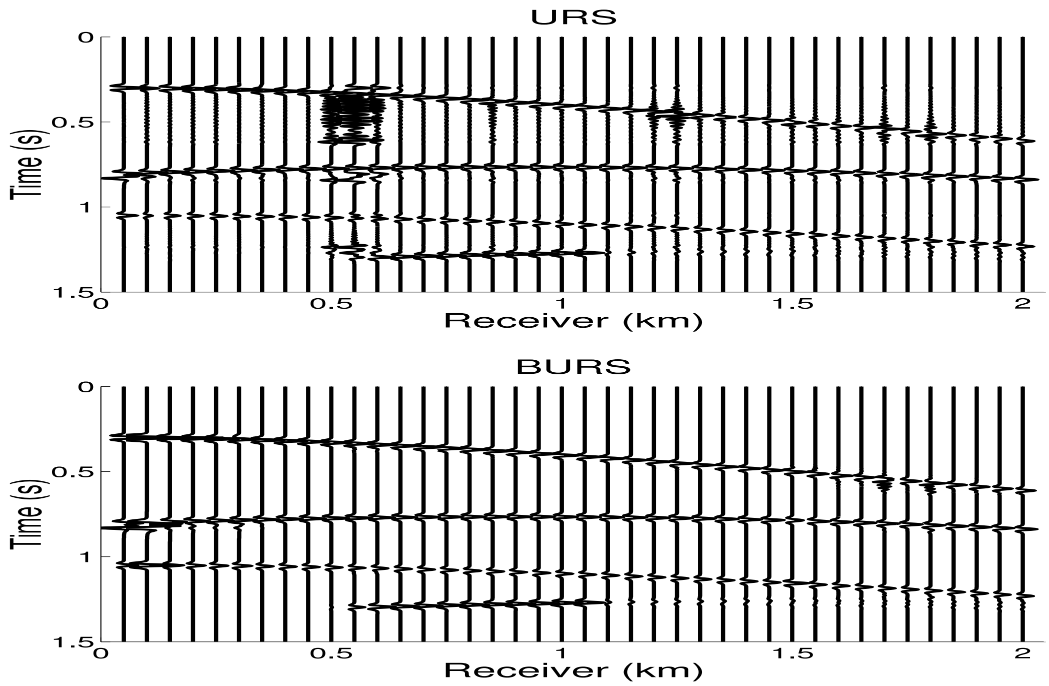

Figure 3: Top: Interpolated seismic section using URS. Bottom: Interpolated seismic section using BURS, $\delta = 0.1$ and $\Delta = 0.2$.

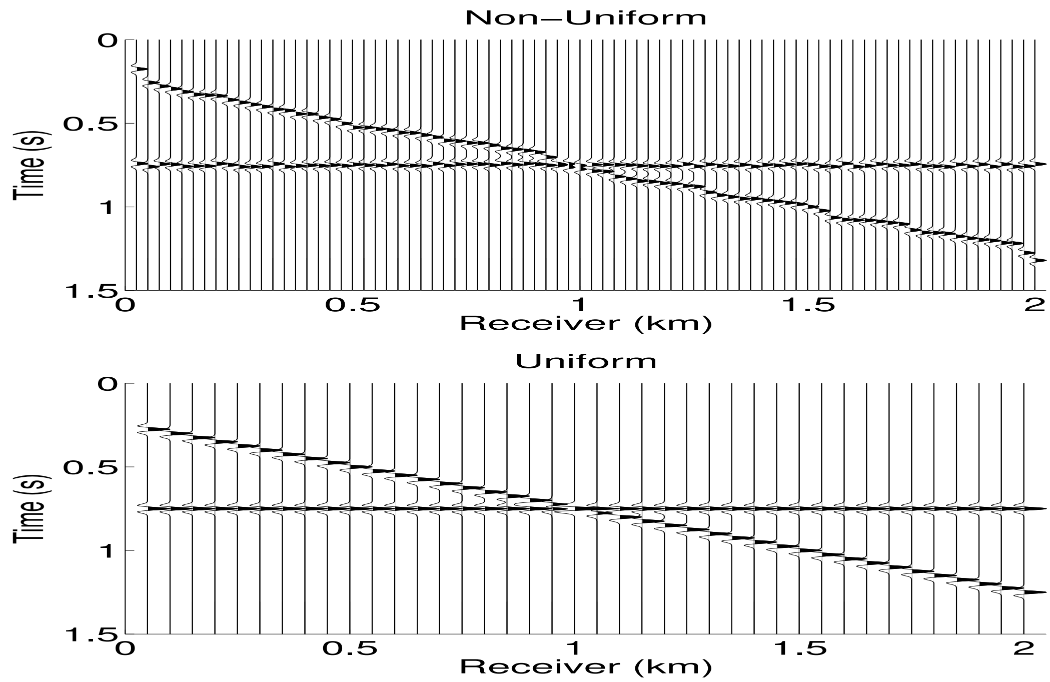

Figure 4: Top: Modeled seismic section for nonequally spaced receiver locations. Bottom: Modeled seismic section for equally spaced receiver locations.

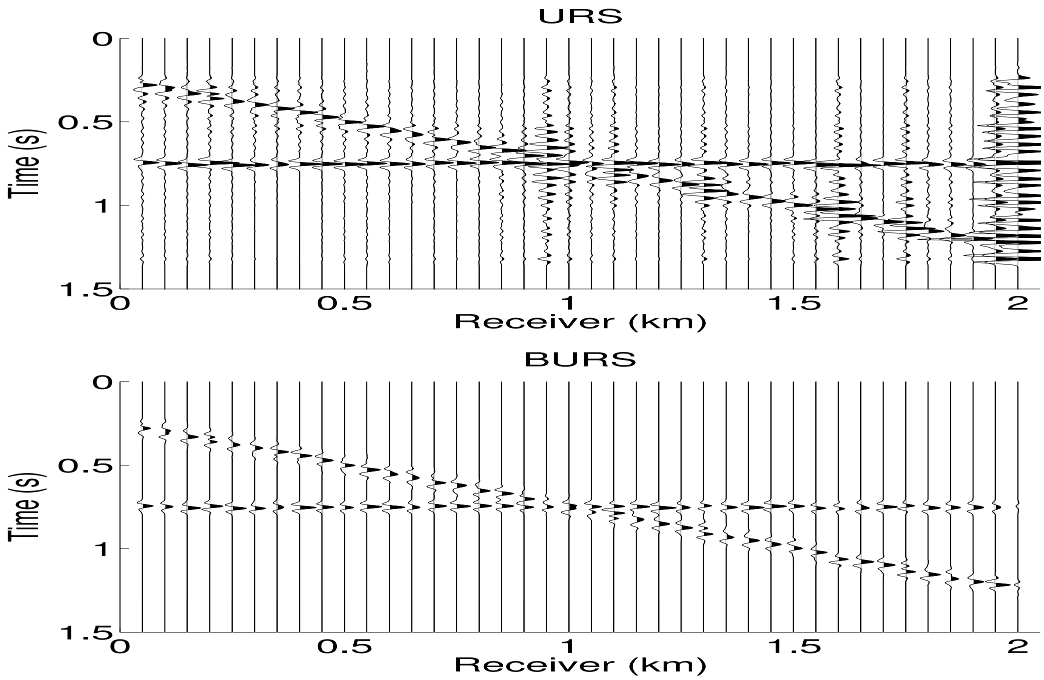

Figure 5: Top: Interpolated seismic section using URS. Bottom: Interpolated seismic section using BURS, $\delta = 0.1$ and $\Delta = 0.2$.

Figure 6: Top: Original Lena Image with 128×128 pixels. Middle: Nonuniform Lena Image with 30% less pixels. Bottom: Reconstructed Lena image using BURS.