Quadratic normal moveouts in isotropic media: a quick tutorial

M. Tygel and L. T. Santos

email: *lucio@ime.unicamp.br*

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ABSTRACT

The design of traveltime moveouts for optimal stacking and inversion of kinematic parameters has always been a subject of much interest in seismic processing. A most prominent role is played by second-order Taylor expressions of reflection traveltimes, commonly known as parabolic and hyperbolic, around a zero-offset (ZO) ray. Here we refer to these expressions as (generalized) quadratic normal moveouts. For a common midpoint (CMP) gather and assuming a planar measurement surface, one has the well-known normal moveout (NMO) that is given in terms of a single parameter, the NMO-velocity. For a supergather of non-symmetrical source and receiver pairs around a fixed ZO point, the simple NMO traveltime is replaced by the generalized quadratic (parabolic or hyperbolic) normal moveout, that depends on three (or eight) parameters in the 2D (or 3D) situation. The simplest expression for the generalized moveout, as used in the CRS method, assumes a planar measurement surface and locally constant near-surface velocities. Corresponding expressions that do not consider these simplifying assumptions, although more complicated, may be required in more complex situations. In this work we present an organized and didactic tutorial on the formulation and derivation of the generalized quadratic normal moveouts in isotropic media.

INTRODUCTION

Traveltime expressions that are able to well approximate reflection events and also convey useful information of such events have always been of key interest in seismic data processing. A common feature to all traveltime formulas, simply referred as moveouts in the seismic literature, is the dependence of a certain number of parameters, that are estimated by means of coherence analysis directly applied on the seismic data. More specifically, the parameters are the ones that maximize the stack energy along the moveout. Besides providing a best-possible stack, the traveltime (or kinematic) parameters play an important role in deriving further imaging information, such as a background velocity model for migration, geometrical-spreading compensation and others.

Of particular importance are the moveouts of rays around the ZO reflection ray. The most familiar of such moveouts, the *Normal Moveout (NMO)*, considers, in its two-dimensional version, a common midpoint (CMP) gather of sources and receivers along a horizontal seismic line. The reflection traveltimes along offset rays not far from the zero-offset (ZO) ray at the CMP are approximated by the one-parameter formula (Dix, 1955)

$$T(h) = \sqrt{T_0^2 + C\, h^2}\ . \tag{1}$$

In the above equation, T is the traveltime from the source to the reflector and back to the receiver, T_0 is the ZO traveltime at the CMP, h is the half-offset between shot and receiver. Finally,

$$C = 4/V_{NMO}^2\ , \tag{2}$$

where V_{NMO} is the NMO-velocity, is the single parameter that is to be inverted from the CMP data. Note that the square of the NMO equation (1) and readily be seen as a second-order Taylor expansion with respect to half-offset.

Under the same conditions as above, a more accurate equation than the NMO equation (1) is the two-parameter *Shifted Hyperbola* formula (de Bazelaire, 1988; Castle, 1994)

$$T(h) = T_0(1 - A) + \sqrt{(A\,T_0)^2 + B\,h^2}\;. \tag{3}$$

The two parameters A and B,that are to be inverted from the CMP data, bear a relationship the previous single NMO-parameter C, namely

$$C = B/A. \tag{4}$$

Also observe that the Shifted Hyperbola equation (3) reduces to the NMO equation (1) if we take $A = 1$.

Still considering the 2D CMP situation, nonhyperbolic moveouts have been introduced to account for larger offsets in isotropic or weakly transversely isotropic media. The moveout expressions are given in the form of a three-parameter continued fractions expression (see, e.g., Tsvankin and Thomsen (2002), Fomel and Gretchka (2001))

$$T^2(h) = T_0^2 + Ch^2 + \frac{Dh^4}{1 + Eh^2}, \tag{5}$$

where C is the NMO parameter. More details on the meaning and properties of the above nonhyperbolic moveout can be found in Fomel and Gretchka (2001) and references therein.

In the last years, more general moveout formulas have been developed, which are not restricted to the CMP configuration and, moreover, take into account a possibly irregular topography at the measurement surface. The point of departure for some of such formulas is to apply a second-order Taylor approximation to the traveltime with respect to the distances of source and receiver from the ZO point. The procedure leads to the so-called parabolic or hyperbolic traveltime moveout, as used, for example in the CRS method. In the following, the general second-order Taylor approximations of the traveltime around the ZO ray will be simply called quadratic normal moveouts.

It is to be mentioned, however, that different approximations, not based on Taylor expansions are also well established. This is the case, for example, of the *Multifocus Moveout* (Gelchinsky et al., 1999; Tygel et al., 1999). In particular, the Shifted-Hyperbola equation (3) is a particular case of such expressions.

An important advantage of Taylor moveouts is their general formulation, also valid for both 2D and 3D situations. Taylor parabolic and hyperbolic moveouts in 3D for arbitrary source and receiver location are well described in Ursin (1982). That paper uses the simple and direct formalism of Taylor expansions to quickly derive and elegant expressions. Under the assumption that sources and receivers lie on given smooth surfaces, the corresponding moveout expressions are obtained in Schleicher et al. (1993) using the surface-to-surface formalism of Bortfeld (1989). The connection to Ursin's expressions are also described in Schleicher et al. (1993). Both papers assume locally constant velocities (that is, negligible velocity gradients) at the source and receiver points. Moreover, the effects of a possibly rugged topography, although implicit in the expressions in Ursin's paper, are not explicitly considered in neither in Ursin (1982) nor in Schleicher et al. (1993).

Using the formalism of paraxial ray theory, which includes ray and surface-to-surface propagator matrices, Cervený (2001) provides a complete treatment of the quadratic (parabolic and hyperbolic) moveouts, without a particular attention to the case of normal moveouts. Although the arguments and results described in Cervený (2001) are, of course, very correct and complete, the presentation demands, perhaps, too much of an ordinary reader, especially those more involved with practical applications.

After Hubral (1983), the appealing and useful concepts of the normal (N) and normal-incident-point (NIP) waves were incorporated in the Taylor formulation of the reflection moveouts in the vicinity of the ZO ray. For instance, for sources and receivers along a horizontal line in the vicinity of a central (ZO) point, the 2D ZO CRS method uses the hyperbolic normal moveout (see, e.g., Tygel et al. (1997),Müller (1999))

$$T^2(m, h) = \left(T_0 + \frac{2\,\sin\beta}{v_0} m_x\right)^2 + \frac{2\,T_0\,\cos^2\beta}{v_0}[K_N\,m^2 + K_{NIP}\,h^2]\;. \tag{6}$$

Here, m and h denote the midpoint (relative to the central point) and half-offset coordinates of the source and receiver pair, T_0 is the ZO traveltime at the central point. Moreover, β, K_N and K_{NIP}, referred to as the ZO CRS parameters, denote the emergence angle of the ZO ray with respect to the surface normal and the curvatures of the N- and NIP-waves, respectively, all these quantities evaluated at the central point.

Finally, v_0 is the medium velocity at the central point. An implicit assumption of the above formula is that the velocity gradient at the central point is negligible. Müller (1999) derives the above formula by comparing the substituting the propagator matrix of a ZO ray in terms of the curvatures K_N and K_{NIP} given in Hubral (1983) into the hyperbolic traveltime expression in Schleicher et al. (1993). It is to be remarked that also the (nonhyperbolic) multifocus moveout with respect to the ZO ray can be also expressed in terms of the ZO CRS parameters, namely (Tygel et al., 1997)

$$T(x_s, x_g) = \frac{R_s}{v_0}\left[\sqrt{1+\frac{2\sin\beta}{R_s}x_s+\frac{x_s^2}{R_s^2}}-1\right]+\frac{R_g}{v_0}\left[\sqrt{1+\frac{2\sin\beta}{R_g}x_g+\frac{x_g^2}{R_g^2}}-1\right], \qquad (7)$$

where x_s and x_g denote the relative distances of the source and receiver to to the ZO point, v_0 and β have the same meaning as in the CRS formula (6) and R_s and R_g are wavefront radii of curvature given by

$$\frac{1}{R_s} = K_s = \frac{1}{1-\gamma}(K_N - \gamma K_{NIP}) \quad \text{and} \quad \frac{1}{R_g} = K_g = \frac{1}{1+\gamma}(K_N + \gamma K_{NIP}), \qquad (8)$$

with the so-called focussing parameter

$$\gamma = \frac{x_g - x_s}{x_g + x_s - (K_{NIP}\sin\beta)x_s x_r}. \qquad (9)$$

Also in this case, the locally constant assumption on the velocity at the ZO point is considered. It is to observed that the original multifocus moveout of Gelchinsky et al. (1999) has been also extended in Gurevich et al. (2002) to account for general topographic effects.

The contribution of velocity gradients and also the effects due to topography at the measurement surface have been considered in Chira et al. (2001) (for smooth topography) and in Zhang et al. (2002) and Zhang (1999) (for rugged topography). All these papers use the surface-to-surface formulation of parabolic and hyperbolic traveltimes around a fixed central ray (not necessarily a ZO ray) as given in Cervený (2001).

Despite their widespread use in many investigations and practical applications, especially in the framework of the ZO CRS method, it is our feeling that the ZO parabolic and hyperbolic moveouts (namely, quadratic normal moveouts) in 2D and 3D still lack a simple and direct exposition and derivation, for example along the lines of Ursin (1982), that accounts for the following generalizations: (a) Consideration of a velocity gradient at the ZO point; (b) full account of topographic effects and (c) explicit dependence on the ZO CRS parameters. This is exactly the purpose of this paper.

In the following, we start with the analysis of 2D Taylor traveltimes around an emerging wavefront, characterized by a given (fixed) ray and a given wavefront curvature at the emergence point of the ray. Next we consider reflection traveltimes and observe that, for arbitrary rays around a fixed ZO ray, the parabolic and hyperbolic moveouts can be fully described in terms of traveltimes around the N- and NIP-wavefronts that refer to the given ZO ray.

The treatment given in this work is restricted to isotropic media. In this situation, rays and wavefronts are perpendicular and only phase velocities are to be considered. Moreover, velocity gradients can be accounted by the change in position only (and not on change of ray direction). The isotropic assumption poses a number of simplifications on the obtained formulas. The extension of the isotropic results, as described here, to the general anisotropic case would be very welcome.

2D TRAVELTIMES AROUND AN EMERGING WAVEFRONT

We consider a fixed (central) ray together with its associated wavefront that emerges at a given point O. Our task is to obtain a Taylor-type approximation of the traveltimes in the vicinity of the reference point O as the wavefront progresses away from it.

Local coordinates

Referring to Figure 1, we consider the local Cartesian coordinate system (ξ, η) with the origin at the emergence point O and ξ-axis lying along the tangent to the wavefront. The η-axis is chosen to point in the direction of wavefront propagation.

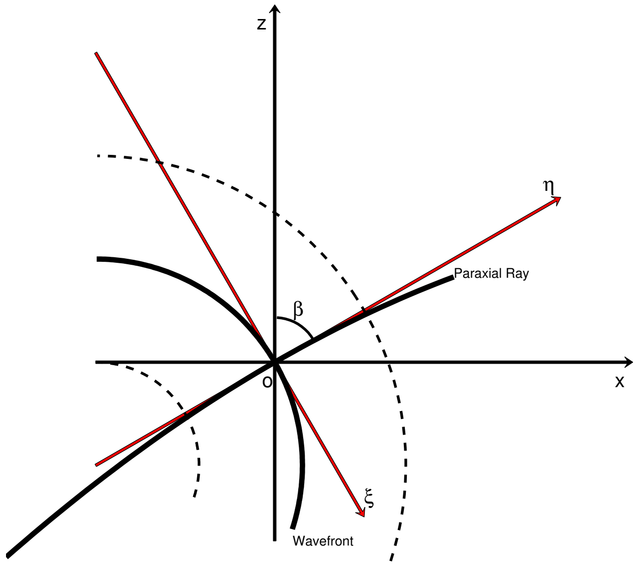

Figure 1: Global and Local Cartesian Coordinate System.

The second-order Taylor approximation for the traveltime, $t(\boldsymbol{\rho})$, at the point $\boldsymbol{\rho} = (\xi, \eta)^T$ in the vicinity of the origin reads

$$t(\boldsymbol{\rho}) = t_0 + \nabla t(\mathbf{0})\boldsymbol{\rho} + \frac{1}{2}\boldsymbol{\rho}^T \nabla^2 t(\mathbf{0})\boldsymbol{\rho}, \tag{10}$$

where

$$t_0 = t(\mathbf{0}), \quad \nabla t(\mathbf{0}) = (t_\xi(\mathbf{0}), t_\eta(\mathbf{0})) \quad \text{and} \quad \nabla^2 t(\mathbf{0}) = \begin{bmatrix} t_{\xi\xi}(\mathbf{0}) & t_{\xi\eta}(\mathbf{0}) \\ t_{\eta\xi}(\mathbf{0}) & t_{\eta\eta}(\mathbf{0}) \end{bmatrix}. \tag{11}$$

To compute the coefficients given by equation (11), we first recall that the traveltime, $t(\boldsymbol{\rho})$, has to satisfy the isotropic eikonal equation

$$|\nabla t|^2 = t_\xi^2 + t_\eta^2 = 1/v^2 \,, \tag{12}$$

al all points $\boldsymbol{\rho}$ under consideration. In particular, because of our choice of the coordinate system, namely the ξ-axis being tangent to the wavefront at the origin and the η-axis pointing in the direction of wavefront propagation, we readily find

$$t_\xi(\mathbf{0}) = 0 \quad \text{and} \quad t_\eta(\mathbf{0}) = 1/v_0 \,, \tag{13}$$

where we denoted $v(\mathbf{0}) = v_0$. To obtain the elements of the Hessian matrix, we differentiate the eikonal equation (12) with respect to ξ and η, respectively, to find

$$t_\xi \; t_{\xi\xi} + t_\eta \; t_{\eta\xi} = -\, v_\xi / v^3 \quad \text{and} \quad t_\xi \; t_{\xi_\eta} + t_\eta \; t_{\eta\eta} = -\, v_\eta / v^3 \,. \tag{14}$$

Setting $\boldsymbol{\rho} = \mathbf{0}$ yields

$$t_{\eta\xi}(\mathbf{0}) = t_{\xi\eta}(\mathbf{0}) = -\, v_\xi^0 / v_0^2 \,, \quad \text{and} \quad t_{\eta\eta}(\mathbf{0}) = -\, v_\eta^0 / v_0^2 \,. \tag{15}$$

with the notation $v_\xi^0 = v_\xi(\mathbf{0})$ and $v_\eta^0 = v_\eta(\mathbf{0})$.

We now show that the remaining element, $t_{\xi\xi}(\mathbf{0})$, has a simple relation to the curvature, K_0, of the wavefront at the origin. To see this, we make use of the fact that the wavefront, being tangent to the ξ-coordinate axis at the origin, admits, near that point the convenient parameterization

$$\eta = \eta(\xi), \tag{16}$$

for which, the wavefront curvature can be expressed as

$$K(\xi) = -\, \frac{\eta''(\xi)}{[1 + (\eta'(\xi))^2]^{3/2}} \,. \tag{17}$$

The reason of the minus signal in the above equation is that we adopt the usual convention of a positive curvature for a concave wavefront in the direction of propagation. Setting $\xi = 0$ in equation (17) yields

$$K_0 = K(0) = -\,\eta''(0). \tag{18}$$

As a next step, we use the identity

$$t(\xi, \eta(\xi)) \equiv t_0\,, \tag{19}$$

that is valid for all points $(\xi, \eta(\xi))$ at the wavefront where the above parameterization holds. Differentiating both sides of equation (19) twice with respect to ξ, we get

$$t_\xi + t_\eta\,\eta_\xi = 0 \quad \text{and} \quad t_{\xi\xi} + 2t_{\eta\xi}\eta_\xi + t_{\eta\eta}(\eta_\xi)^2 + t_\eta\,\eta_{\xi\xi} = 0. \tag{20}$$

Setting $\xi = 0$ in the above equation and also under the consideration of equation (13) yields the well-known result

$$t_{\xi\xi}(\mathbf{0}) = -\,\frac{1}{v_0}\,\eta_{\xi\xi} = K_0/v_0. \tag{21}$$

Substituting equations (13), (15) and (21) into equation (10), we obtain the second-order Taylor or parabolic traveltime,

$$t(\xi, \eta) = t_0 + \frac{\eta}{v_0} + \frac{K_0}{2v_0}\,\xi^2 + \frac{1}{2}\boldsymbol{\rho}^T\mathbf{E}\boldsymbol{\rho}\,, \tag{22}$$

where

$$\mathbf{E} = -\frac{1}{v_0^2}\begin{bmatrix} 0 & v_\xi^0 \\ v_\xi^0 & v_\eta^0 \end{bmatrix}. \tag{23}$$

The last term of the above equation, that accounts for the contribution due to the velocity gradient at the emergence point of the central ray, will be referred as the *inhomogeneity term*.

We finally observe that, for observation points on the ξ-axis, $\boldsymbol{\rho} = (\xi, 0)$, we obtain the simplest formula

$$t(\xi, 0) = t_0 + \frac{K_0}{2v_0}\,\xi^2, \tag{24}$$

which does not depend on the velocity gradients.

Global Coordinates

The previously considered local Cartesian (ξ, η)-system will now be changed to a global Cartesian (x, z)-system. This is certainly the real situation, since the wavefront is, in principle, not known. That unknown angle will become a parameter in the new formula. The relationship between the new (global) and old (local) Cartesian coordinate systems is simply a rotation about the emergence angle, β, of the normal to the wavefront at O with respect to the new z-axis (see Figure 1). Setting $\mathbf{r} = (x, z)^T$, the corresponding coordinate transformation is given, in matrix form, as

$$\mathbf{r} = \mathbf{G}\boldsymbol{\rho}, \quad \text{with} \quad \mathbf{G} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}, \tag{25}$$

from which, by the orthogonality property, $\mathbf{G}^{-1} = \mathbf{G}^T$, of the matrix $\mathbf{G}$, and an application of the chain rule of derivatives, we find

$$\boldsymbol{\rho} = \mathbf{G}^T\mathbf{r}, \quad \text{and} \quad \begin{bmatrix} v_\xi \\ v_\eta \end{bmatrix} = \mathbf{G}^T \begin{bmatrix} v_x \\ v_z \end{bmatrix}. \tag{26}$$

Substituting the above relations into equation (22), we arrive at the moveout expression in global coordinates

$$t(x, z) = t_0 + \frac{1}{v_0}[x\ \sin\beta + z\ \cos\beta] + \frac{K_0}{2\,v_0}[x\ \cos\beta - z\ \sin\beta]^2 + \frac{1}{2}\,\mathbf{r}^T\ \mathbf{B}\ \mathbf{r}\,, \tag{27}$$

where the matrix $\mathbf{B}$ that appears in the inhomogeneity term is given by

$$\mathbf{B} = \mathbf{G}\,\mathbf{E}\,\mathbf{G}^T = -\frac{1}{v_0^2}\begin{bmatrix} a & c \\ c & b \end{bmatrix}, \tag{28}$$

with

$$\begin{aligned} a &= \sin\beta\,[v_x^0\,(1+\cos^2\beta) - v_z^0\,\cos\beta\sin\beta]\,, \\ b &= \cos\beta\,[v_z^0\,(1+\sin^2\beta) - v_x^0\,\cos\beta\sin\beta]\,, \\ c &= v_x^0\,\cos^3\beta + v_z^0\,\sin^3\beta\,. \end{aligned} \tag{29}$$

It is important to note that the matrix $\mathbf{E}$ has been also tranformed into global coordinates using relations (26).

We finally observe that the parameters a, b and c in equation (30) all depend on the velocity gradient at the origin. Therefore, for the case of a locally constant velocity at the origin, i.e., $v_x^0 = v_z^0 = 0$, all these parameters vanish, leading to the reduced expression

$$t(x,z) = t_0 + \frac{1}{v_0}[x\,\sin\beta + z\,\cos\beta] + \frac{K_0}{2v_0}[x\,\cos\beta - z\,\sin\beta]^2\,. \tag{30}$$

2D TAYLOR REFLECTION MOVEOUTS AROUND THE ZO RAY

We now consider, still in the 2D situation, reflected rays from arbitrary source and receiver locations around a fixed ZO reference ray. Assuming a fixed global Cartesian coordinate system, we consider, without loss of generality, that the (fixed) ZO ray departs and emerges from the origin of that system. To make full use of the symmetries that are attached to the ZO ray, we adopt, as usual done in the literature, midpoint and half-offset coordinates $\mathbf{m} = (m_x, m_z)$ and $\mathbf{h} = (h_x, h_z)$, to locate a source and receiver pair around the ZO ray. In other words if $\mathbf{r}_s = (x_s, z_s)$ and $\mathbf{r}_g = (x_g, z_g)$ denote the global Cartesian coordinates of the source and receiver, respectively, the corresponding midpoint and half-offset coordinates, $(\mathbf{m}, \mathbf{h})$, satisfy the relations

$$\mathbf{m} = (\mathbf{r}_g + \mathbf{r}_s)/2 \quad \text{and} \quad \mathbf{h} = (\mathbf{r}_g - \mathbf{r}_s)/2. \tag{31}$$

The parabolic moveout (namely, the second-order Taylor approximation of the traveltime), now denoted by $T(\mathbf{m}, \mathbf{h})$, around the ZO traveltime, $T_0 = T(\mathbf{0}, \mathbf{0})$ reads

$$T(\mathbf{m},\mathbf{h}) = T_0 + \boldsymbol{\nabla} T(\mathbf{0})\,(\mathbf{m},\mathbf{h})^T + \frac{1}{2}\,(\mathbf{m},\mathbf{h})\,\boldsymbol{\nabla}^2 T(\mathbf{0})\,(\mathbf{m},\mathbf{h})^T, \tag{32}$$

where

$$\boldsymbol{\nabla} T(\mathbf{0}) = \left(\frac{\partial T}{\partial \mathbf{m}}, \frac{\partial T}{\partial \mathbf{h}}\right) \quad \text{and} \quad \boldsymbol{\nabla}^2 T(\mathbf{0}) = \begin{bmatrix} \dfrac{\partial^2 T}{\partial \mathbf{m}^2} & \dfrac{\partial^2 T}{\partial \mathbf{m}\partial \mathbf{h}} \\ \dfrac{\partial^2 T}{\partial \mathbf{h}\partial \mathbf{m}} & \dfrac{\partial^2 T}{\partial \mathbf{h}^2} \end{bmatrix} \tag{33}$$

with the notations

$$\frac{\partial T}{\partial \mathbf{m}} = \left(\frac{\partial T}{\partial m_x}, \frac{\partial T}{\partial m_z}\right), \quad \frac{\partial T}{\partial \mathbf{h}} = \left(\frac{\partial T}{\partial h_x}, \frac{\partial T}{\partial h_z}\right), \tag{34}$$

and

$$\frac{\partial^2 T}{\partial \mathbf{m}^2} = \left[\frac{\partial^2 T}{\partial m_p m_q}\right], \quad \frac{\partial^2 T}{\partial \mathbf{m}\partial \mathbf{h}} = \left[\frac{\partial^2 T}{\partial m_p h_q}\right], \quad \frac{\partial^2 T}{\partial \mathbf{h}^2} = \left[\frac{\partial^2 T}{\partial h_p h_q}\right], \quad (p,q = x,z), \tag{35}$$

all the above partial derivatives being evaluated at $\mathbf{m} = \mathbf{h} = \mathbf{0}$.

We now observe the fundamental fact that, due to reciprocity, we have, for any coordinate pair $(\mathbf{m}, \mathbf{h})$,

$$T(\mathbf{m}, -\mathbf{h}) = T(\mathbf{m}, \mathbf{h})\,, \tag{36}$$

namely, the traveltime is an odd function of half-offset. As a consequence, in the present ZO situation,

$$\frac{\partial T}{\partial \mathbf{h}} = \frac{\partial^2 T}{\partial \mathbf{m}\partial \mathbf{h}} = \frac{\partial^2 T}{\partial \mathbf{h}\partial \mathbf{m}} = 0, \tag{37}$$

which allows for the appealing traveltime decoupling, characteristic of the ZO situation,

$$T(\mathbf{m},\mathbf{h}) = T(\mathbf{m},\mathbf{0}) + T(\mathbf{0},\mathbf{h}) - T_0. \tag{38}$$

It is easy to recognize that the traveltimes $T(\mathbf{m},\mathbf{0})$ and $T(\mathbf{0},\mathbf{h})$ have obvious meanings, namely as the ZO moveout, $T_{ZO}(\mathbf{m})$, at midpoint $\mathbf{m}$ and as the CMP moveout, $T_{CMP}(\mathbf{h})$, at half-offset $\mathbf{h}$ with respect to the midpoint at the origin. Under the above notation, the parabolic moveout (38) can be recast as

$$T(\mathbf{m},\mathbf{h}) = T_{ZO}(\mathbf{m}) + T_{CMP}(\mathbf{h}) - T_0. \tag{39}$$

Our task, now, is to find suitable independent expressions for the ZO and CMP moveouts. As shown below, these expressions can be easily derived from the results of the previous section upon the introduction of the N- and NIP-waves of Hubral (1983).

- ZO Case: The N-Wave. The ZO moveout, $T_{ZO}(\mathbf{m})$, can be readily interpreted to belong to a wavefront that coincides with the reflector at zero time and progresses towards the measurement surface with half the medium velocity. As explained in Hubral (1983), this hypothetical, that realizes the hypothetical exploding reflector experiment, is the N-wave. As a consequence, the sought for ZO traveltime, $T_{ZO}(\mathbf{m})$, can be readily obtained from equation (27) by just considering twice that traveltime setting $\mathbf{r} = \mathbf{m}^T$ and $K_0 = K_N$, the wavefront curvature of the N-wave at at $\mathbf{0}$. We find

$$\begin{aligned} T_{CMP}(\mathbf{m}) = 2t(\mathbf{m}) = T_0 \quad &+ \quad \frac{2}{v_0}[m_x \ \sin\beta + m_z \ \cos\beta] \\ &+ \quad \frac{K_N}{v_0}[m_x \ \cos\beta - m_z \ \sin\beta]^2 \\ &+ \quad \mathbf{m}\,\mathbf{B}\,\mathbf{m}^T \ , \end{aligned} \tag{40}$$

  where we have considered the fact that $T_0 = 2t_0$.

- CMP Case: The NIP-Wave. The CMP moveout, $T_{CMP}(\mathbf{h})$ can be also be obtained by the results of the previous section upon the introduction of NIP-wave and also taking into account the NIP-wave theorem of Hubral (1983). The NIP-wave theorem states that, up to the second-order Taylor approximation, the CMP traveltime equals the diffraction traveltime at NIP. As a consequence, the CMP traveltime can be considered as the traveltime sum along the rays that connect the source, at $-\mathbf{h}$ and the receiver, at $\mathbf{h}^T$ to a "diffraction point" at NIP. Both these traveltimes can be accounted for using the theory of the previous section, upon the consideration of the NIP-wave, that starts at time zero as a point source at NIP and progresses to the measurement surface at half the velocity of the medium. Setting $K_0 = K_{NIP}$ in equation (27) and considering $\mathbf{r} = -\mathbf{h}^T$ and $\mathbf{r} = \mathbf{h}^T$, we readily find

$$\begin{aligned} T_{CMP}(\mathbf{h}) = t(-\mathbf{h}) + t(\mathbf{h}) = T_0 \quad &+ \quad \frac{K_{NIP}}{v_0}[h_x \ \cos\beta - h_z \ \sin\beta]^2 \\ &+ \quad \mathbf{h}\,\mathbf{B}\,\mathbf{h}^T \ , \end{aligned} \tag{41}$$

  where we have, once more, considered the relation $T_0 = 2t_0$.

Putting together equations (39), (40) and (41), we arrive at the *parabolic* approximation for the reflection traveltime, namely

$$\begin{aligned} T(\mathbf{m},\mathbf{h}) = T_0 \quad &+ \quad \frac{2}{v_0}[m_x \ \sin\beta + m_z \ \cos\beta] \\ &+ \quad \frac{K_N}{v_0}[m_x \ \cos\beta - m_z \ \sin\beta]^2 + \frac{K_{NIP}}{v_0}[h_x \ \cos\beta - h_z \ \sin\beta]^2 \\ &+ \quad \mathbf{m}\,\mathbf{B}\,\mathbf{m}^T + \mathbf{h}\,\mathbf{B}\,\mathbf{h}^T \ . \end{aligned} \tag{42}$$

The corresponding *hyperbolic* approximation, that is, the second-order Taylor formula for T^2, can be readily obtained by squaring both sides of the parabolic traveltime (42) and discarding the higher-order terms. We find,

$$\begin{aligned} T^2(\mathbf{m},\mathbf{h}) &= \left(T_0 + \frac{2}{v_0}[m_x\ \sin\beta + m_z\ \cos\beta]\right)^2 \\ &+ \frac{2\,T_0\,K_N}{v_0}[m_x\ \cos\beta - m_z\ \sin\beta]^2 + \frac{2\,T_0\,K_{NIP}}{v_0}[h_x\ \cos\beta - h_z\ \sin\beta]^2 \\ &+ 2\,T_0\,[\mathbf{m}\,\mathbf{B}\,\mathbf{m}^T + \mathbf{h}\,\mathbf{B}\,\mathbf{h}^T]\ . \end{aligned} \tag{43}$$

2D CRS traveltime

It is interesting to consider the particular case of source and receiver at the surface $z = 0$ and a locally constant velocity at the origin. This is obtained by just setting in equation (43), $v_x^0 = v_z^0 = 0$, as well as $\mathbf{m} = (m, 0)$ and $\mathbf{h} = (h, 0)$, leading to

$$T^2(m,h) = \left(T_0 + \frac{2\ \sin\beta}{v_0}m\right)^2 + \frac{2\,T_0\ \cos^2\beta}{v_0}[K_N\ m^2 + K_{NIP}\ h^2]\ . \tag{44}$$

Equation (44) is the one that is commonly used for stacking and parameter estimations in the CRS method. As already mentioned, we readily observe that, in the CMP configuration, $m = 0$, the CRS formula (44) reduces to Dix's NMO moveout,

$$T^2(h) = T_0^2 + 4\,h^2/V_{NMO}^2\ , \tag{45}$$

where $V_{NMO}^2 = 2\,v_0/(T_0\,K_{NIP}\ \cos^2\beta)$.

Remark: For inversion purposes, the general traveltime formula (43) can, in principle, be used as a parametric surface for stacking and inversion. In this case, we have six attributes to be determined: the emergence angle β, the wavefront curvatures K_N and K_{NIP}, and the gradient velocity parameters a, b and c. If we consider, as done usually by the CRS method, a locally-constant velocity, i.e., $\mathbf{B} = \mathbf{0}$, the number of parameters reduces to three.

EXTENSION TO THE 3–D SITUATION

The previous analysis can be easily extended to the three-dimensional case. In the same way as before, we start the analysis with the consideration of traveltimes around a given ray together and its wavefront that emerge at point O at the measurement surface.

Local coordinates

The local (ξ, μ, η)-Cartesian system in which the (ξ, μ)-plane is tangent to the wavefront at the origin and the η-axis points to the propagation direction is now considered. To facilitate the natural comparison with the previous 2D case, we now consider $\boldsymbol{\rho} = (\xi, \mu, \eta)^T$. The second-order Taylor expansion of traveltime in 3D has the same form of its 2D counterpart of equation (10), namely

$$t(\boldsymbol{\rho}) = t_0 + \nabla t(\mathbf{0})\boldsymbol{\rho} + \frac{1}{2}\boldsymbol{\rho}^T\nabla^2 t(\mathbf{0})\boldsymbol{\rho}, \tag{46}$$

where now,

$$\nabla t(\mathbf{0}) = (t_\xi(\mathbf{0}), t_\mu(\mathbf{0}), t_\eta(\mathbf{0})) \quad \text{and} \quad \nabla^2 t(\mathbf{0}) = \begin{bmatrix} t_{\xi\xi}(\mathbf{0}) & t_{\xi\mu}(\mathbf{0}) & t_{\xi\eta}(\mathbf{0}) \\ t_{\mu\xi}(\mathbf{0}) & t_{\mu\mu}(\mathbf{0}) & t_{\mu\eta}(\mathbf{0}) \\ t_{\eta\xi}(\mathbf{0}) & t_{\eta\mu}(\mathbf{0}) & t_{\eta\eta}(\mathbf{0}) \end{bmatrix}. \tag{47}$$

The isotropic eikonal equation, also valid for any point around the wavefront, can be written as (compare with equation (12))

$$|\nabla t|^2 = t_\xi^2 + t_\mu^2 + t_\eta^2 = 1/v^2\ , \tag{48}$$

where v is the velocity field. Therefore, in analogy to the previous 2D case, we have, from the chosen coordinate system,

$$t_\xi(\mathbf{0}) = t_\mu(\mathbf{0}) = 0 \quad \text{and} \quad t_\eta(\mathbf{0}) = 1/v_0 \ . \tag{49}$$

In analogy to the 2D case, differentiation of the eikonal equation (48) with respect to ξ_1 and ξ_2 and evaluation at the origin yields (compare with equation (15))

$$t_{\eta\xi}(\mathbf{0}) = t_{\xi\eta}(\mathbf{0}) = -\, v_\xi^0/v_0^2 \ , \quad t_{\eta\mu}(\mathbf{0}) = t_{\mu\eta}(\mathbf{0}) = -\, v_\mu^0/v_0^2 \quad \text{and} \quad t_{\eta\eta}(\mathbf{0}) = -\, v_\eta^0/v_0^2 \ . \tag{50}$$

Still following the 2D case, we parameterize the wavefront in the vicinity of the origin as (compare with equation (16))

$$\eta = \eta(\xi, \mu), \tag{51}$$

for which the curvature matrix at the origin point is given by

$$\mathbf{K}^0 = \mathbf{K}(\mathbf{0}) = -\begin{bmatrix} \eta_{\xi\xi}^0 & \eta_{\xi\mu}^0 \\ \eta_{\mu\xi}^0 & \eta_{\mu\mu}^0 \end{bmatrix}. \tag{52}$$

Observe that the same signal convention for the wavefront curvature (positive for concave in the propagation direction) have been adopted. Upon twice partial differentiation of the wavefront identity (compare with equation (19))

$$t(\xi, \mu, \eta(\xi, \mu)) \equiv t_0, \tag{53}$$

with respect to ξ and μ, we can relate the upper left 2×2 submatrix of the traveltime Hessian at the origin by the formula (compare with equation (21))

$$t_{pq}(\mathbf{0}) = -\frac{1}{v_0}\eta_{pq}^0 = \frac{1}{v_0}K_{pq}^0. \tag{54}$$

with $p, q = \xi, \mu$. Putting together all the above results, we arrive at

$$t(\xi, \mu, \eta) = t_0 + \frac{\eta}{v_0} + \frac{1}{2\, v_0}(\xi, \mu)\mathbf{K}^0(\xi, \mu)^T + \frac{1}{2}\boldsymbol{\rho}^T\mathbf{E}\boldsymbol{\rho} \ , \tag{55}$$

where $\mathbf{E}$ is given in the 3–D case by

$$\mathbf{E} = -\frac{1}{v_0^2}\begin{bmatrix} 0 & 0 & v_\xi^0 \\ 0 & 0 & v_\mu^0 \\ v_\xi^0 & v_\mu^0 & v_\eta^0 \end{bmatrix}. \tag{56}$$

Global Coordinates

Still parallel to the 2D case, we now change from the local Cartesian (ξ, μ, η)-system to a global Cartesian (x, y, z)-global coordinate system. The new system is obtained by a cascaded rotation of an angle β that transforms the η-axis into the z-axis followed by a rotation of angle α that takes the (transformed) ξ-axis into the x-axis. Setting $\mathbf{r} = (x, y, z)^T$, the transformation can be given in matrix form as (compare with equation (25))

$$\mathbf{r} = \mathbf{G}\boldsymbol{\rho} \quad \text{with} \quad \mathbf{G} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha & \cos\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha & -\sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \tag{57}$$

where the matrix $\mathbf{G}$ is a product of two matrix components

$$\mathbf{G} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}. \tag{58}$$

>From right to left, the first matrix represents a rotation of angle β around the μ axis until the η-axis and z-axis coincide and the second matrix is a further rotation of angle α around the z-axis. After these two rotations, the system (ξ, μ, η) coincides with the system (x, y, z).

Substituting equation (57) into equation (55), we obtain, after some linear algebra, the 3D traveltime in global coordinates (compare with equation (27))

$$
\begin{aligned}
t(x,y,z) = t_0 \quad &+ \quad \frac{1}{v_0}[x\ \cos\alpha\ \sin\beta - y\ \sin\alpha\ \sin\beta + z\ \cos\beta] \\
&+ \quad \frac{K_{11}}{2\ v_0}[x\ \cos\alpha\ \cos\beta - y\ \sin\alpha\ \cos\beta - z\ \sin\beta]^2 \\
&+ \quad \frac{K_{12}}{v_0}[x\ \cos\alpha\ \cos\beta - y\ \sin\alpha\ \cos\beta - z\ \sin\beta]\ [x\ \sin\alpha + y\ \cos\alpha] \\
&+ \quad \frac{K_{22}}{2\ v_0}[x\ \sin\alpha + y\ \cos\alpha]^2 \\
&+ \quad \frac{1}{2}\mathbf{r}^T\mathbf{B}\mathbf{r}\ , \qquad (59)
\end{aligned}
$$

where,

$$
\mathbf{B} = \mathbf{G}\ \mathbf{E}\ \mathbf{G}^T = -\frac{1}{v_0^2}\begin{bmatrix} a & d & c \\ d & e & f \\ c & f & b \end{bmatrix}, \qquad (60)
$$

with

$$
\begin{aligned}
a &= \cos\alpha\sin\beta\ [v_x^0\ (2-\cos^2\alpha\sin^2\beta) + v_y^0\cos\alpha\sin\alpha\sin^2\beta - v_z^0\cos\alpha\sin\beta\cos\beta], \\
b &= \cos\beta\ [-v_x^0\cos\alpha\sin\beta\cos\beta + v_y^0\sin\alpha\sin\beta\cos\beta + v_z^0\ (2-\cos^2\beta)], \\
c &= v_x^0\cos\beta(1-\cos^2\alpha\sin^2\beta) + v_y^0\sin\alpha\cos\alpha\sin^2\beta\cos\beta + v_z^0\cos\alpha\sin^3\beta, \\
d &= -v_x^0\sin\alpha\sin\beta(1-\cos^2\alpha\sin^2\beta) + v_y^0\cos\alpha\sin\beta(1-\sin^2\alpha\sin^2\beta) + v_z^0\sin\alpha\cos\alpha\sin^2\beta\cos\beta, \\
e &= \sin\alpha\sin\beta\ [-v_x^0\sin\alpha\cos\alpha\sin^2\beta - v_y^0(2-\sin^2\alpha\sin^2\beta) - v_z^0\sin\alpha\sin\beta\cos\beta], \\
f &= v_x^0\sin\alpha\cos\alpha\sin^2\beta\cos\beta + v_y^0\cos\beta(1-\sin^2\alpha\sin^2\beta) - v_z^0\sin\alpha\sin^3\beta\ . \qquad (61)
\end{aligned}
$$

As in the 2-D case, for a locally constant velocity at the origin, $\mathbf{B} = \mathbf{0}$ and then, the last term of equation (59) vanishes. Moreover, if $\alpha = 0$, formula (59) reduces to formula (27) in the xz-plane ($y = 0$).

Reflection Traveltime

The same analysis for 2D case can be now applied for the 3-D case. For a general source and receiver pair, $(\mathbf{r}_s, \mathbf{r}_g)$, in 3D space around the the origin, we consider the 3D midpoint and half-offset coordinates

$$
\mathbf{m} = (m_x, m_y, m_z) = (\mathbf{r}_g + \mathbf{r}_s)/2 \quad \text{and} \quad \mathbf{h} = (h_x, h_y, h_z) = (\mathbf{r}_g - \mathbf{r}_s)/2. \qquad (62)
$$

The reflection traveltime by $T(\mathbf{m},\mathbf{h})$ can be readily obtained by applying equation (59) conveniently to approximate the traveltimes $T(\mathbf{m},\mathbf{0}) = 2\ t(\mathbf{m})$ and $T(\mathbf{0},\mathbf{h}) = t(-\mathbf{h}) + t(\mathbf{h})$. We then find the 3D parabolic moveout and, after squaring, the hyperbolic traveltime. For simplicity, we only write the hyperbolic one,

$$
\begin{aligned}
T^2(\mathbf{m},\mathbf{h}) &= \left(T_0 + \frac{2}{v_0}[m_x\ \cos\alpha\ \sin\beta - m_y\ \sin\alpha\ \sin\beta + m_z\ \cos\beta]\right)^2 \\
&+ \frac{2\,T_0\,K_{11}^N}{v_0}[m_x\ \cos\alpha\ \cos\beta - m_y\ \sin\alpha\ \cos\beta - m_z\ \sin\beta]^2 \\
&+ \frac{4\,T_0\,K_{12}^N}{v_0}[m_x\ \cos\alpha\ \cos\beta - m_y\ \sin\alpha\ \cos\beta - m_z\ \sin\beta]\ [m_x\ \sin\alpha + m_y\ \cos\alpha] \\
&+ \frac{2\,T_0\,K_{22}^N}{v_0}[m_x\ \sin\alpha + m_y\ \cos\alpha]^2 \\
&+ \frac{2\,T_0\,K_{11}^{NIP}}{v_0}[h_x\ \cos\alpha\ \cos\beta - h_y\ \sin\alpha\ \cos\beta - h_z\ \sin\beta]^2 \\
&+ \frac{4\,T_0\,K_{12}^{NIP}}{v_0}[h_x\ \cos\alpha\ \cos\beta - h_y\ \sin\alpha\ \cos\beta - h_z\ \sin\beta]\ [h_x\ \sin\alpha + h_y\ \cos\alpha] \\
&+ \frac{2\,T_0\,K_{22}^{NIP}}{v_0}[h_x\ \sin\alpha + h_y\ \cos\alpha]^2 \\
&+ 2\,T_0\,[\mathbf{m}\,\mathbf{B}\,\mathbf{m}^T + \mathbf{h}\,\mathbf{B}\,\mathbf{h}^T]\ . \qquad (63)
\end{aligned}
$$

where $T_0 = T(\mathbf{0},\mathbf{0}) = 2\,t(\mathbf{0})$, and $\mathbf{B}$ is given by equations (60) and (61).

Remark: The traveltime formula (63) can also be used as a parametric surface for inversion purposes. The number of attributes now has been increased to eleven: two emergence angles α and β, six wavefront curvatures (three for $\mathbf{K}^N$ and three for $\mathbf{K}^{NIP}$, and the three components of the velocity gradient. As before, the number of parameters is reduced for locally-constant velocity (that is, when the velocity gradient is negligible). In this case, the number of parameters to be inverted reduces to eight.

SUMMARY AND CONCLUSIONS

Taylor-type moveouts, especially the second-order parabolic and hyperbolic are routinely used for stacking and inversion purposes in the processing of seismic data. Of special relevance are the traveltimes around the ZO ray, simply called here quadratic normal moveouts, for which a number of useful symmetries and simplifications are valid. In this paper we have provided an organized presentation, discussion and derivation of the quadratic normal moveouts in isotropic media, using the simplest possible mathematical framework. In this sense, we have followed the appealing approach of Ursin (1982) with the inclusion of the generalizations: (a) Consideration of a velocity gradient at the ZO point; (b) full account of topographic effects and (c) explicit dependence on the ZO CRS parameters.

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