A Tutorial on Elliptical Anisotropy

Claudia Vanelle

email: *vanelle@dkrz.de*

keywords: *elliptical anisotropy, reflection coefficients, traveltimes, geometrical spreading*

ABSTRACT

This paper gives a short summary of the properties of anisotropic media with elliptical symmetry. It was motivated by the need for analytic expressions for the evaluation and verification of related computer algorithms. After a brief introduction and derivation of the phase and ray (group) velocities and the polarisation vectors I give expressions for the plane wave reflection and transmission coefficients at a boundary between two elliptically anisotropic half-spaces. These are followed by expressions for the traveltimes and the geometrical spreading for homogeneous media with elliptical anisotropy. Please note, that the resulting expressions are equally valid for isotropic media if the elastic coefficients are chosen accordingly. A final short description of my computer codes for the calculation of these quantities concludes the paper.

INTRODUCTION

A medium with elliptical anisotropy and a vertical symmetry axis is characterised by the density-normalised elasticity tensor ($A_{ik} = C_{ik}/\rho$)

$$\underline{\underline{\mathsf{A}}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & & & \\ & A_{11} & A_{13} & & & \\ & & A_{33} & & & \\ & & & A_{44} & & \\ & & & & A_{44} & \\ & & & & & A_{66} \end{pmatrix} \tag{1}$$

with the additional constraints

$$\begin{aligned} A_{12} &= A_{11} - 2\,A_{66} \\ (A_{13} + A_{44})^2 &= (A_{11} - A_{44})(A_{33} - A_{44}) \quad . \end{aligned} \tag{2}$$

Let the slowness vector be denoted by $\mathbf{p}$. Since $\underline{\underline{\mathsf{A}}}$ displays rotational symmetry with respect to the vertical (z- or 3-) axis I choose $\mathbf{p}$ in a way that $p_y = p_2 = 0$ and

$$\mathbf{p} = \left(\frac{\sin\phi}{V},\, 0,\, \frac{\cos\phi}{V} \right) \quad , \tag{3}$$

where ϕ is the phase angle made by $\mathbf{p}$ and the vertical (z- or 3-) axis, and V is the phase velocity. I introduce the abbreviations d_{11} and d_{33} with

$$d_{11} = A_{11} - A_{44} \quad \text{and} \quad d_{33} = A_{33} - A_{44} \quad . \tag{4}$$

This leads to the following non-vanishing elements of the Christoffel matrix $\Gamma_{ik} = a_{ijkl} p_j p_l$:

$$
\begin{aligned}
\Gamma_{11} &= A_{11}\, p_1^2 + A_{44}\, p_3^2 \;=\; A_{11}\frac{\sin^2\phi}{V^2} + A_{44}\frac{\cos^2\phi}{V^2} \quad ,\\
\Gamma_{22} &= A_{66}\, p_1^2 + A_{44}\, p_3^2 \;=\; A_{66}\frac{\sin^2\phi}{V^2} + A_{44}\frac{\cos^2\phi}{V^2} \quad ,\\
\Gamma_{33} &= A_{44}\, p_1^2 + A_{33}\, p_3^2 \;=\; A_{44}\frac{\sin^2\phi}{V^2} + A_{33}\frac{\cos^2\phi}{V^2} \quad ,\\
\Gamma_{13} &= (A_{13} + A_{44})\, p_1\, p_3 \;=\; \sqrt{d_{11} d_{33}}\,\frac{\sin\phi\,\cos\phi}{V^2} \quad .
\end{aligned}
\tag{5}
$$

PHASE VELOCITIES

The solution of the Christoffel equation for the displacement vector $\mathbf{u}$,

$$(\Gamma_{ik} - \delta_{ik}) u_k = 0 \quad , \tag{6}$$

where δ_{ik} is Kronecker's delta, requires that

$$\left|\Gamma_{ik} - G^{(n)}\delta_{ik}\right| = 0 \quad . \tag{7}$$

This determinant leads to the characteristic polynome of third order, whose three solutions are the eigenvalues $G^{(n)} = 1$, with $n = 1, 2, 3$. I define the index 1 to be a *q*SV wave, 2 an SH wave, and 3 a *q*P wave. The physical meaning of these definitions will become apparent in the next section on the polarisation vectors. For simplicity, the indices are abbreviated by SV, SH, and P, omitting the *q*.

Insertion of Γ_{ik} for the elliptic case yields three phase velocities $V^{(n)}$:

$$
\begin{aligned}
V^{SV} &= \sqrt{A_{44}} \quad ,\\
V^{SH} &= \sqrt{A_{66}\sin^2\phi^{SH} + A_{44}\cos^2\phi^{SH}} \quad ,\\
V^{P} &= \sqrt{A_{11}\sin^2\phi^{P} + A_{33}\cos^2\phi^{P}} \quad .
\end{aligned}
\tag{8}
$$

POLARISATION

The three eigenvectors $\mathbf{g}^{(n)}$ that obey

$$(\Gamma_{ik} - \delta_{ik}) g_k^{(n)} = 0 \tag{9}$$

are the polarisation vectors of the three waves with the phase velocities $V^{(n)}$. The polarisations are given by

$$
\begin{aligned}
\mathbf{g}^{SV} &= \left(m^{SV}\cos\phi^{SV},\, 0,\, -l^{SV}\sin\phi^{SV}\right) \quad ,\\
\mathbf{g}^{SH} &= (0,\, 1,\, 0) \quad ,\\
\mathbf{g}^{P} &= \left(l^{P}\sin\phi^{P},\, 0,\, m^{P}\cos\phi^{P}\right) \quad ,
\end{aligned}
\tag{10}
$$

where the abbreviations $l^{(n)}$ and $m^{(n)}$ are introduced:

$$
\begin{aligned}
l^{(n)} &= \sqrt{\frac{d_{11}}{d_{11}\sin^2\phi^{(n)} + d_{33}\cos^2\phi^{(n)}}} \quad ,\\
m^{(n)} &= \sqrt{\frac{d_{33}}{d_{11}\sin^2\phi^{(n)} + d_{33}\cos^2\phi^{(n)}}} \quad .
\end{aligned}
\tag{11}
$$

The signs of the polarisation vectors are chosen in a way that the $\mathbf{g}^{(n)}$ form an orthonormal system, see Figure 1. The wave associated with index 3 is a *q*P wave, which I have abbreviated with P for shortness. The index 1 corresponds to a quasi shear wave, abbreviated with SV. The SH wave has index 2.

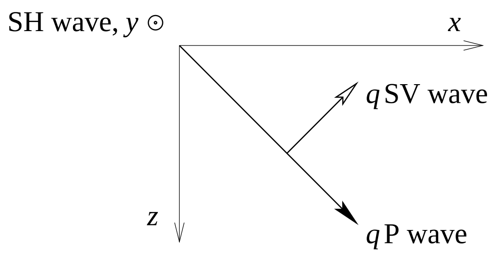

Figure 1: Polarisation vectors in a medium with elliptical anisotropy. The *q*P and *q*SV waves propagate in the x-z plane, the SH wave polarisation vector is oriented along the y axis, pointing to the reader.

RAY (GROUP) VELOCITIES

The components of the ray or group velocity vectors for the three wave types, $\mathbf{v}^{(n)}$ (denoted by lower case letters to distinguish the group velocities from the phase velocities $V^{(n)}$) are given by

$$v_i^{(n)} = a_{ijkl}\, g_j^{(n)}\, g_k^{(n)}\, p_l^{(n)} \quad , \tag{12}$$

leading to

$$\begin{aligned} \mathbf{v}^{SV} &= \left(\sqrt{A_{44}}\sin\phi^{SV},\, 0,\, \sqrt{A_{44}}\cos\phi^{SV}\right) \quad , \\ \mathbf{v}^{SH} &= \left(\frac{A_{66}}{V^{SH}}\sin\phi^{SH},\, 0,\, \frac{A_{44}}{V^{SH}}\cos\phi^{SH}\right) \quad , \\ \mathbf{v}^{P} &= \left(\frac{A_{11}}{V^{P}}\sin\phi^{P},\, 0,\, \frac{A_{33}}{V^{P}}\cos\phi^{P}\right) \quad . \end{aligned} \tag{13}$$

Introducing the ray angle $\theta^{(n)}$ with $\tan\theta^{(n)} = v_x^{(n)}/v_z^{(n)}$ yields

$$\begin{aligned} \tan\theta^{SV} &= \tan\phi^{SV} \quad , \\ \tan\theta^{SH} &= \frac{A_{66}}{A_{44}}\tan\phi^{SH} \quad , \\ \tan\theta^{P} &= \frac{A_{11}}{A_{33}}\tan\phi^{P} \quad , \end{aligned} \tag{14}$$

and

$$\begin{aligned} v^{SV} &= \sqrt{A_{44}} = V^{SV} \quad , \\ v^{SH} &= \frac{\sqrt{A_{66}^2\sin^2\phi^{SH} + A_{44}^2\cos^2\phi^{SH}}}{V^{SH}} = \left[\frac{\sin^2\theta^{SH}}{A_{66}} + \frac{\cos^2\theta^{SH}}{A_{44}}\right]^{-\frac{1}{2}} \quad , \\ v^{P} &= \frac{\sqrt{A_{11}^2\sin^2\phi^{P} + A_{33}^2\cos^2\phi^{P}}}{V^{P}} = \left[\frac{\sin^2\theta^{P}}{A_{11}} + \frac{\cos^2\theta^{P}}{A_{33}}\right]^{-\frac{1}{2}} \quad . \end{aligned} \tag{15}$$

REFLECTION AND TRANSMISSION COEFFICIENTS

The displacement vector for a plane wave of type n is expressed by

$$\mathbf{u}^{(n)} = U^{(n)}\,\mathbf{g}^{(n)}\,\mathrm{e}^{-i\omega(t-\tau^{(n)})} \quad . \tag{16}$$

where $U^{(n)}$ is the scalar amplitude associated with the wavetype n. The eikonal or phase function $\tau^{(n)}$ is

$$\tau^{(n)} = \nabla\tau^{(n)}\cdot\mathbf{r} = \mathbf{p}^{(n)}\cdot\mathbf{r} \quad . \tag{17}$$

Consider now a plane boundary between two homogeneous elliptically anisotropic half spaces at the depth $z = 0$ (see Figure 2). Depending on the type of the incident wave, reflected and transmitted waves of different types are generated. An incident SH wave leads to reflected and transmitted SH waves, whereas in the cases of incident *q*SV or *q*P waves conversion from *q*SV to *q*P and vice versa can also occur. Therefore, an incident *q*SV wave will generate not only reflected and transmitted *q*SV waves, but also reflected and transmitted *q*P waves. The same applies to an incident *q*P wave which will lead to reflected and transmitted *q*SV and *q*P waves. Each of these waves can be written in terms of Equation (16). In addition to the upper index n for the wavetype, the individual displacement vectors will be denoted with the lower index 0 for the incident wave, R for the reflected wave, and T for the transmitted wave.

To determine the reflection and transmission coefficients of the displacement, the following boundary conditions must be fulfilled:

$$\begin{aligned}
&\text{Continuity of displacement} &:\quad & u_x, u_y, u_z \quad ,\\
&\text{Continuity of shear stress} &:\quad & \sigma_{xz} = \rho A_{44}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \quad ,\\
& & & \sigma_{yz} = \rho A_{44}\frac{\partial u_y}{\partial z} \quad ,\\
&\text{Continuity of normal stress} &:\quad & \sigma_{zz} = \rho\left(A_{13}\frac{\partial u_x}{\partial x} + A_{33}\frac{\partial u_z}{\partial z}\right) \quad .
\end{aligned}$$

The spatial derivatives of the displacement components are

$$\frac{\partial u_i}{\partial x_j} = i\omega\, \mathrm{e}^{-i\omega(t-\mathbf{p}\cdot\mathbf{r})}\, u_i\, p_j \quad . \tag{18}$$

The slowness vectors of the incident, reflected, and transmitted waves are given by

$$\begin{aligned}
&\text{incident wave} &:\quad \mathbf{p}_0^{(n)} &= \frac{1}{V_0^{(n)}}\,(\sin\phi_0^{(n)},\,0,\,\cos\phi_0^{(n)}) \quad ,\\
&\text{reflected wave} &:\quad \mathbf{p}_R^{(n)} &= \frac{1}{V_R^{(n)}}\,(\sin\phi_R^{(n)},\,0,\,-\cos\phi_R^{(n)}) \quad ,\\
&\text{transmitted wave} &:\quad \mathbf{p}_T^{(n)} &= \frac{1}{V_T^{(n)}}\,(\sin\phi_T^{(n)},\,0,\,\cos\phi_T^{(n)}) \quad .
\end{aligned} \tag{19}$$

Snell's law requires that the horizontal slowness $p = \sin\phi/V$ remains constant. This can lead to imaginary angles ϕ. In that case the displacement given by Equation (16) will show exponential behaviour along z. To avoid an increase in amplitude, the cosine of the angle ϕ must be either a real or a positive imaginary number.

Application of the boundary conditions and phase matching leads to equations for the reflection and transmission coefficients. Since the SH wave is decoupled from the *q*P and *q*SV waves, the SH and *q*P/*q*SV cases can be treated separately.

SH waves

Continuity of the y component of the displacement, u_y, and the shear stress σ_{yz} leads to

$$\begin{aligned}
R_{SH-SH} &= \frac{U_R^{SH}}{U_0^{SH}} = \frac{\rho^{(1)}\,A_{44}^{(1)}\,V_2\,\cos\phi_1 - \rho^{(2)}\,A_{44}^{(2)}\,V_1\,\cos\phi_2}{\rho^{(1)}\,A_{44}^{(1)}\,V_2\,\cos\phi_1 + \rho^{(2)}\,A_{44}^{(2)}\,V_1\,\cos\phi_2} \quad ,\\
T_{SH-SH} &= \frac{U_T^{SH}}{U_0^{SH}} = \frac{2\,\rho^{(1)}\,A_{44}^{(1)}\,V_2\,\cos\phi_1}{\rho^{(1)}\,A_{44}^{(1)}\,V_2\,\cos\phi_1 + \rho^{(2)}\,A_{44}^{(2)}\,V_1\,\cos\phi_2} \quad ,
\end{aligned} \tag{20}$$

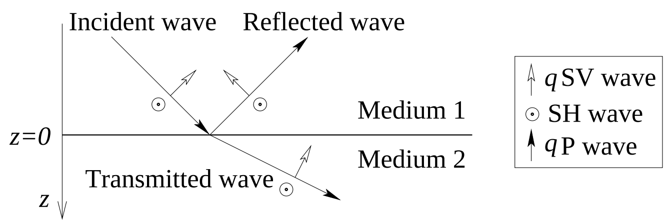

Figure 2: Boundary between two homogeneous elliptical media and orientation of the polarisation vectors of the incident, reflected, and transmitted waves.

where index 1 describes the properties of medium 1 (with the incident/reflected wave) and index 2 those of medium 2 with the transmitted wave. The reflection angle, ϕ_R^{SH} is equal to the incidence angle $\phi_0^{SH} = \phi_1$, and the transmission angle $\phi_T^{SH} = \phi_2$ can be computed from Snell's law, leading to

$$\phi_T^{SH} = \arctan\left[\frac{A_{44}^{(2)}}{\frac{1}{p^2} - A_{66}^{(2)}}\right]^{\frac{1}{2}} \quad , \tag{21}$$

where p can be computed from the quantities of the incident wave, i.e. $p = \sin\phi_0^{SH}/V_0^{SH}$.

*q*P-*q*SV waves

Continuity of the x and z components of the displacement vectors, the shear stress σ_{xz} and the normal stress σ_{zz} leads to two linear system of equations with four unknowns, one system for an incident SV wave,

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} \begin{pmatrix} R_{SV-P} \\ T_{SV-P} \\ R_{SV-SV} \\ T_{SV-SV} \end{pmatrix} = \begin{pmatrix} Y_1^{SV} \\ Y_2^{SV} \\ Y_3^{SV} \\ Y_4^{SV} \end{pmatrix} \quad , \tag{22}$$

and a second for an incident P wave:

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} \begin{pmatrix} R_{P-P} \\ T_{P-P} \\ R_{P-SV} \\ T_{P-SV} \end{pmatrix} = \begin{pmatrix} Y_1^{P} \\ Y_2^{P} \\ Y_3^{P} \\ Y_4^{P} \end{pmatrix} \quad . \tag{23}$$

As for the SH case, the reflection coefficients $R_{nn'}$ and transmission coefficients $T_{nn'}$ of the displacement are given by the amplitude ratio between the reflected/transmitted wave (of type n) and the incident wave (of type n'):

$$R_{nn'} = \frac{U_R^n}{U_0^{n'}} \quad \text{and} \quad T_{nn'} = \frac{U_T^n}{U_0^{n'}} \quad . \tag{24}$$

The 4×4 matrix $\underline{\underline{X}}$ is the same in both Equations, (22) and (23). Its elements are

$$
\begin{aligned}
X_{11} &= U_R^P\, l_R^P\, \sin\phi_R^P \quad , \\
X_{12} &= -\, U_T^P\, l_T^P\, \sin\phi_T^P \quad , \\
X_{13} &= -\, U_R^{SV}\, m_R^{SV}\, \cos\phi_R^{SV} \quad , \\
X_{14} &= -\, U_T^{SV}\, m_T^{SV}\, \cos\phi_T^{SV} \quad , \\
\\
X_{21} &= -\, U_R^P\, m_R^P\, \cos\phi_R^P \quad , \\
X_{22} &= -\, U_T^P\, m_T^P\, \cos\phi_T^P \quad , \\
X_{23} &= -\, U_R^{SV}\, l_R^{SV}\, \sin\phi_R^{SV} \quad , \\
X_{24} &= U_T^{SV}\, l_T^{SV}\, \sin\phi_T^{SV} \quad , \\
\\
X_{31} &= -\, \frac{U_R^P}{V_R^P}\, C_{55}^{(1)}\, \sin\phi_R^P\, \cos\phi_R^P\, (l_R^P + m_R^P) \quad , \\
X_{32} &= -\, \frac{U_T^P}{V_T^P}\, C_{55}^{(2)}\, \sin\phi_T^P\, \cos\phi_T^P\, (l_T^P + m_T^P) \quad , \\
X_{33} &= -\, \frac{U_R^{SV}}{V_R^{SV}}\, C_{55}^{(1)}\, (l_R^{SV} \sin^2\phi_R^{SV} - m_R^{SV} \cos^2\phi_R^{SV}) \quad , \\
X_{34} &= \frac{U_T^{SV}}{V_T^{SV}}\, C_{55}^{(2)}\, (l_T^{SV} \sin^2\phi_T^{SV} - m_T^{SV} \cos^2\phi_T^{SV}) \quad , \\
\\
X_{41} &= \frac{U_R^P}{V_R^P}\, (C_{13}^{(1)}\, {}_R^P\, \sin^2\phi_R^P + C_{33}^{(1)}\, m_R^P\, \cos^2\phi_R^P) \quad , \\
X_{42} &= -\, \frac{U_T^P}{V_T^P}\, (C_{13}^{(2)}\, {}_T^P\, \sin^2\phi_T^P + C_{33}^{(2)}\, m_T^P\, \cos^2\phi_T^P) \quad , \\
X_{43} &= -\, \frac{U_R^{SV}}{V_R^{SV}}\, \sin\phi_R^{SV}\, \cos\phi_R^{SV}\, (C_{13}^{(1)}\, m_R^{SV} - C_{33}^{(1)}\, l_R^{SV}) \quad , \\
X_{44} &= -\, \frac{U_T^{SV}}{V_T^{SV}}\, \sin\phi_T^{SV}\, \cos\phi_T^{SV}\, (C_{13}^{(2)}\, m_T^{SV} - C_{33}^{(2)}\, l_T^{SV}) \quad . \qquad (25)
\end{aligned}
$$

The right hand sides of Equations (22) and (23) are given by

$$
\begin{aligned}
Y_1^{SV} &= -\, U_0^{SV}\, m_0^{SV}\, \cos\phi_0^{SV} \quad , \\
Y_2^{SV} &= U_0^{SV}\, l_0^{SV}\, \sin\phi_0^{SV} \quad , \\
Y_3^{SV} &= \frac{U_0^{SV}}{V_0^{SV}}\, C_{55}^{(1)}\, (l_0^{SV} \sin^2\phi_0^{SV} - m_0^{SV} \cos^2\phi_0^{SV}) \quad , \\
Y_4^{SV} &= -\, \frac{U_0^{SV}}{V_0^{SV}}\, \sin\phi_0^{SV}\, \cos\phi_0^{SV}\, (C_{13}^{(1)}\, m_0^{SV} - C_{33}^{(1)}\, l_0^{SV}) \quad . \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
Y_1^P &= -\, U_0^P\, l_0^P\, \sin\phi_0^P \quad , \\
Y_2^P &= -\, U_0^P\, m_0^P\, \cos\phi_0^P \quad , \\
Y_3^P &= -\, \frac{U_0^P}{V_0^P}\, C_{55}^{(1)}\, \sin\phi_0^P\, \cos\phi_0^P\, (l_0^P + m_0^P) \quad , \\
Y_4^P &= -\, \frac{U_0^P}{V_0^P}\, (C_{13}^{(1)}\, {}_0^P\, \sin^2\phi_0^P + C_{33}^{(1)}\, m_0^P\, \cos^2\phi_0^P) \quad . \qquad (27)
\end{aligned}
$$

The reflection and transmission angles are again determined from Snell's law:

$$\begin{aligned}
\phi_R^P &= \arctan\left[\frac{A_{33}^{(1)}}{\frac{1}{p^2}-A_{11}^{(1)}}\right]^{\frac{1}{2}} \quad ,\\
\phi_T^P &= \arctan\left[\frac{A_{33}^{(2)}}{\frac{1}{p^2}-A_{11}^{(2)}}\right]^{\frac{1}{2}} \quad ,\\
\phi_R^{SV} &= \arcsin(p\,V_R^{SV}) \quad ,\\
\phi_T^{SV} &= \arcsin(p\,V_T^{SV}) \quad ,
\end{aligned} \tag{28}$$

where again, $p = \sin\phi_0^{(n)}/V_0^{(n)}$ is computed from the incident wave with n equal to P or SV.

Equations (22) and (23) can be solved for the individual coefficients by the usual methods for systems of linear equations.

Normalised R/T coefficients

Another possibility is to express the coefficients normalised with respect to the energy flux perpendicular to the interface. The normalised reflection and transmission coefficients $\mathcal{R}_{nn'}$ and $\mathcal{T}_{nn'}$ are obtained from the standard coefficients $R_{nn'}$ and $T_{nn'}$ by

$$\begin{aligned}
\mathcal{R}_{nn'} &= \left|\frac{\rho_R\,v_R^{n'}\,\cos\phi_R^{n'}}{\rho_0\,v_0^n\,\cos\phi_0^n}\right|^{\frac{1}{2}} R_{nn'} \quad ,\\
\mathcal{T}_{nn'} &= \left|\frac{\rho_T\,v_T^{n'}\,\cos\phi_T^{n'}}{\rho_0\,v_0^n\,\cos\phi_0^n}\right|^{\frac{1}{2}} T_{nn'} \quad .
\end{aligned} \tag{29}$$

The wavetype denoted by n' is again that of the incident wave, index n corresponds to the reflected or transmitted wave.

TRAVELTIMES

Consider a homogeneous medium with the vector $\mathbf{r} = (x,y,z) = (g_x - s_x, g_y - s_y, g_z - s_z)$ describing the distance between the source (**s**) and receiver (**g**) positions, and its modulus, $r = \sqrt{x^2+y^2+z^2}$. The traveltime $\tau^{(n)}$ of a wave of type n propagating from the source to the receiver is given by $\tau^{(n)} = r/v^{(n)}$. This results in the following traveltimes:

$$\begin{aligned}
\tau^{SV} &= \sqrt{\frac{x^2+y^2+z^2}{A_{44}}} \quad ,\\
\tau^{SH} &= \sqrt{\frac{x^2+y^2}{A_{66}}+\frac{z^2}{A_{44}}} \quad ,\\
\tau^{P} &= \sqrt{\frac{x^2+y^2}{A_{11}}+\frac{z^2}{A_{33}}} \quad .
\end{aligned} \tag{30}$$

GEOMETRICAL SPREADING

The relative geometrical spreading $L^{(n)}$ that a wave of type n undergoes in a homogeneous medium can be expressed by

$$L^{(n)} = \frac{\cos\theta^{(n)}}{\sqrt{\left|\det\underline{\mathsf{N}}^{(n)}\right|}}\,\frac{v^{(n)}}{V^{(n)}} \quad , \tag{31}$$

where the 2×2 matrix $\underline{\mathsf{N}}^{(n)}$ is the second-order mixed derivative matrix of the traveltimes $\tau^{(n)}$ with respect to the source and receiver positions:

$$N_{IJ}^{(n)} = -\frac{\partial^2 \tau^{(n)}}{\partial s_I \, \partial g_J} \quad . \tag{32}$$

(Indices I and J take the values 1 and 2.) Differentiation of Equation (30) leads to the following expressions for the relative geometrical spreading:

$$\begin{aligned} L^{SV} &= \sqrt{A_{44}}\, r \quad , \\ L^{SH} &= \frac{\sqrt{A_{44}}\, A_{66}}{v\, V}\, r \quad , \\ L^{P} &= \frac{A_{11}\, \sqrt{A_{33}}}{v\, V}\, r \quad . \end{aligned} \tag{33}$$

COMPUTER PROGRAMS

This section gives a short description of the computer codes that make use of the results from the previous sections. The programs are free software under the GNU public license and can be obtained from the author.

Reflection and Transmission Coefficients

The program `elli_coef.f` computes the elastic standard reflection and transmission coefficients as described above. The linear systems (22) and (23) are solved with Kramer's method, where the determinants are computed analytically. This program comes with the graphical user interface (GUI) `elli_coef.pl`. After entering the elastic parameters of the two media and specifying the incident and outgoing wavetypes, the program computes (the `Apply` button) or computes and displays (the `Plot` button) the desired reflection or transmission coefficient. The SU (seismic unix) routines `ftnstrip` and `xgraph` are required to display the coefficients.

Traveltimes and Spreading

The program `elli_ttl.c` computes traveltimes and geometrical spreading in three dimensions for a homogeneous elliptical medium. The required elastic parameters A_{11}, A_{33}, A_{44}, and A_{66} are given in the input coordinate system with rotational symmetry around the vertical axis. Three angles can be specified to transform the elasticity tensor to a system with arbitrary orientation of the symmetry axis. To do so, the input coordinate system is first rotated around the x axis by the angle `r_x`, then around the y axis by `r_y`, and, finally, around the z axis by `r_z`.

The program computes spreading and traveltimes from Equations (30) and (33) for arbitrary wave type, grid sizes and spacings, and source positions (within the specified volume). The user is prompted for all of these informations after starting the program. Of course, it is also possible to apply I/O-redirect with an input file.

ACKNOWLEDGEMENTS

This work was partly supported by the sponsors of the Wave Inversion Technology (WIT) consortium and the German Research Foundation (DFG, grant Va 207/3-1). Continuous discussions with the members of the Applied Geophysics Group Hamburg are appreciated. I am grateful to Dirk Gajewski and Svetlana Soukina for critical reading and valuable suggestions.

REFERENCES

Červený, V. (2001). *Seismic Ray Theory*. Cambridge University Press.

Dellinger, J. A. (1991). *Anisotropic Seismic Wave Propagation*. PhD thesis, Stanford University.

Schleicher, J., Tygel, M., Ursin, B., and Bleistein, N. (2001). The Kirchhoff-Helmholtz integral for anisotropic elastic media. *Wave Motion*, pages 353–364.

Vanelle, C. and Gajewski, D. (2002). Determining geometrical spreading from traveltimes in anisotropic media. In *Annual WIT Report*, volume 6.