Impedance-type approximations of the P–P elastic reflection coefficient: Modelling and AVO inversion

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ABSTRACT

The normal-incidence elastic compressional reflection coefficient admits an exact, simple expression in terms of the acoustic impedance, namely, the product of the P-wave velocity and density, at both sides of the interface. With slight modifications a similar expression can, also exactly, express the oblique-incidence acoustic reflection coefficient. A severe limitation on the use of the above two reflection coefficients in analyzing seismic reflection data is that they provide no information on shear-wave velocities that refer to the interface. In this paper, we address the natural question of whether a suitable impedance concept can be introduced for which arbitrary P-P reflection coefficients can be expressed in an analogous form as their counterpart acoustic ones. We formulate this problem by considering the mathematical conditions to be satisfied by such a general impedance function. Although no closed-form exact solution exists, our analysis provides a general framework for which, under suitable restrictions of the medium parameters, possible impedance functions can be derived. In particular, the well-established concept of elastic impedance and the recently introduced concept of reflection impedance can be better understood. Concerning these two impedances, we examine their potential for modelling and for the estimation of the AVO indicators of intercept and gradient. For typical synthetical examples, we show that the reflection impedance formulation provides consistently better results than those obtained using the elastic impedance.

INTRODUCTION

Estimation of reflection coefficients from primary reflections is one of the key objectives of seismic amplitude analysis. In elastic media, reflection coefficients have a rather complicated dependence to the medium parameters (P- and S- wave velocities and density) at both sides of the interface. As a consequence, even if the reflection coefficients are correctly estimated from the seismic data, inversion of the medium parameters using the full formulas is to avoided. To overcome these difficulties, geophysicists have tried to express reflection coefficients in terms of quantities that, on one side, can be estimated from the data and, on the other hand, provide a better access to the medium parameters. Following the simple cases of normal-incidence in elastic media or generally oblique incidence in acoustic media, the reflection coefficient can be easily expressed by simple formulas involving the acoustic impedance, namely the product between the P-wave velocity and the density. The acoustic impedance fulfills both previously indicated requirements, namely, it carries direct information about the medium parameters and, moreover, provides a simple expression for the reflection coefficient. As shown below, the attractive simplicity of the above expressions cannot, unfortunately, be fully extended to elastic oblique-incidence. Nevertheless, under suitable restrictions of the medium parameters, convenient impedance definitions can be introduced to provide useful approximations of the elastic reflection coefficients.

The first of these impedance concepts is that of *elastic impedance* as introduced by Connolly (1999), under the assumption of a constant ratio, $K = \beta^2/\alpha^2$, between the square of the S- and P-wave velocities of the media. A discussion on the formulation and practical use of the elastic impedance concept is given in Whitcombe (2002). A second impedance concept, called *reflection impedance*, has been recently introduced in Santos et al. (2002). It is based on the alternative condition that the shear-wave velocity is related to the density as $\rho = b\beta^\gamma$. In other words, a "Gardner's type" law, generally considered for compressional velocities, is also assumed for shear velocities.

In this work, we analyze the problem of finding an impedance function for the general elastic P-P reflection coefficient. In the framework of the analysis, we review the concepts of elastic and reflection impedance available in the literature. We finally consider the potential of the two impedance concepts for modelling the reflection coefficient, as well as on the extraction of AVO indicators such as the intercept and gradient. Based on simple, but typical, synthetic experiments, we conclude that the reflection impedance is able to more accurately perform both tasks, as compared to the elastic impedance.

SIMPLE CASES

We start by considering the simple cases of normal incidence in elastic/acoustic media and oblique-incidence in acoustic media, in which the reflection coefficient has an attractive simple expression in terms of an impedance function.

Normal incidence in elastic/acoustic media

The compressional wave reflection coefficient for normal incidence is given by

$$R_0(\rho_i, \alpha_i) = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1} \ , \tag{1}$$

where ρ_i and α_i denote the density and P-velocity, respectively, at the incident side $(i = 1)$ and at the opposite side $(i = 2)$ of the reflecting interface. Note that the normal-incidence reflection coefficient given by equation (1) is independent of the S-wave velocities of the two media. Introducing the *acoustic impedance*

$$AI = \rho\ \alpha \ , \tag{2}$$

namely the product of the density, ρ, with the P-wave velocity, α, the reflection coefficient, R_0, can be recast in the simple form

$$R_0(\rho_i, \alpha_i) = \frac{AI_2 - AI_1}{AI_2 + AI_1} \ . \tag{3}$$

Oblique incidence in acoustic media

In the case of non-normal incidence in acoustic media (S-wave velocity $\beta = 0$), the corresponding reflection coefficient is given by

$$R_a(\rho_i, \alpha_i, \theta) = \frac{AI_2\ \sec\theta_2 - AI_1\ \sec\theta_1}{AI_2\ \sec\theta_2 + AI_1\ \sec\theta_1} \ , \tag{4}$$

where AI_i is the acoustic impedance as in equation (3), together with the additional requirements (Snell's law)

$$\theta_1 = \theta \quad \text{and} \quad \frac{\sin\theta_1}{\alpha_1} = \frac{\sin\theta_2}{\alpha_2} \ . \tag{5}$$

Defining the *angular acoustic impedance*

$$AI(\theta) = \rho\alpha\sec\theta = AI\sec\theta \ , \tag{6}$$

the expression for the oblique-incidence acoustic reflection coefficient is given by an expression similar to equation (3), namely,

$$R_a(\rho_i, \alpha_i, \theta) = \frac{AI_2(\theta_2) - AI_1(\theta_1)}{AI_2(\theta_2) + AI_1(\theta_1)} \ . \tag{7}$$

Note that

$$AI(\theta = 0) = AI \quad \text{and} \quad R_a(\rho_i, \alpha_i, \theta = 0) = R(\rho_i, \alpha_i) \ , \tag{8}$$

as expected.

GENERAL CASE

For the general oblique incidence in elastic media (S-wave velocity $\beta \neq 0$), the expression for the P-P reflection coefficient is also the ratio between two quantities,

$$R = \frac{P[\rho_i, \alpha_i, \beta_i, \theta]}{Q[\rho_i, \alpha_i, \beta_i, \theta]} \; . \tag{9}$$

However, the numerator, P, and denominator, Q, do not have the simple form as the previous ones (see, e.g., Aki and Richards (1980)).

As seen by the recent literature (see, e.g., Connolly (1999); Mallick (2001)), it makes sense to look for a quantity (impedance) $I \equiv I(\rho, \alpha, \beta, \theta)$ for which the reflection coefficient can be given, at least approximately, by an expression of the form

$$R = \frac{I_2 - I_1}{I_2 + I_1} \; . \tag{10}$$

To examine this interesting question, we find useful to introduce the concept of the *reflectivity function*, as defined below.

The reflectivity function

Roughly speaking, the reflectivity function is a measure of the variation of the reflection coefficient as we move along a ray within a layered media. To quantitatively express this variation, we consider that the elastic characteristics, ρ, α and β, as well as the incident angle, θ, are functions of a single variable, σ, that parameterize the ray. This variable can be, e.g., depth or time. In other words, we consider, along the ray, the vector quantity $\eta(\sigma) = (\rho(\sigma), \alpha(\sigma), \beta(\sigma), \theta(\sigma))$. With this understanding, we can recast the reflection coefficient, as given by equation (9), in the form

$$R \equiv R(\sigma, \Delta\sigma) = \frac{P[\eta(\sigma), \eta(\sigma + \Delta\sigma)]}{Q[\eta(\sigma), \eta(\sigma + \Delta\sigma)]} \; . \tag{11}$$

where $\Delta\sigma$ is the parameter increment, chosen to be sufficiently small. In the above formula σ and $\sigma + \Delta\sigma$ replace indices 1 and 2, respectively. For example, $\rho(\sigma)$ replaces ρ_1, $\alpha(\sigma + \Delta\sigma)$ replaces α_2, etc.

The P-P *elastic reflectivity* function $\mathcal{R}$ can be defined as the limit,

$$\mathcal{R}(\sigma) = \lim_{\Delta\sigma \to 0} \frac{R(\sigma, \Delta\sigma)}{\Delta\sigma} \; . \tag{12}$$

Using the *exact* formula for the P-P elastic coefficient (see Aki and Richards (1980)), we readily obtain the expression

$$\mathcal{R}(\sigma) = \frac{1}{2}\Big[1 - 4\beta^2 p^2\Big]\frac{\rho'}{\rho} + \frac{1}{2}\Big[\frac{1}{1 - \alpha^2 p^2}\Big]\frac{\alpha'}{\alpha} - \Big[4\beta^2 p^2\Big]\frac{\beta'}{\beta} \; , \tag{13}$$

where the prime denotes derivative with respect to to σ and p is the ray parameter given by Snell's law

$$p = \frac{\sin\theta(\sigma)}{\alpha(\sigma)} = \frac{\sin\theta(\sigma + \Delta\sigma)}{\alpha(\sigma + \Delta\sigma)} \; . \tag{14}$$

Under the assumption of a flat-layered medium, the ray parameter, is assumed to be constant along the ray. Recall, however, that the angle, θ, is dependent on the parameter σ.

Using the reflectivity function definition (12), and approximating the derivatives in equation (13) by their corresponding discrete differences, i.e., $f' \approx \Delta f/\Delta\sigma$, we arrive at the well-known first-order approximation for R (Aki and Richards (1980)),

$$R \approx \mathcal{R}(\sigma)\;\Delta\sigma \;\approx \frac{1}{2}\Big[1 - 4\frac{\beta^2}{\alpha^2}\sin^2\theta\Big]\frac{\Delta\rho}{\rho} + \frac{1}{2}\Big[\sec^2\theta\Big]\frac{\Delta\alpha}{\alpha} - \Big[4\frac{\beta^2}{\alpha^2}\sin^2\theta\Big]\frac{\Delta\beta}{\beta} \; . \tag{15}$$

The attributes intercept and gradient

For sufficiently small incidence angles, $\tan^2\theta \approx \sin^2\theta$, and then we may rewrite equation (15) as the well-known Intercept and Gradient formula given in Shuey (1985), namely

$$R \approx A + B\sin^2\theta\ , \tag{16}$$

where

$$A = \frac{1}{2}\Big[\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\Big]\ , \text{ and } B = \frac{1}{2}\frac{\Delta\alpha}{\alpha} - 2\frac{\beta^2}{\alpha^2}\Big[\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\Big]\ . \tag{17}$$

The impedance equation

The problem of finding a function I satisfying equation (10) *exactly* is equivalent to that of determining a solution of the differential equation resulting from the computation of the limit in (12), assuming the desired form (10):

$$\mathcal{R}(\sigma) = \lim_{\Delta\sigma\to 0}\left[\frac{1}{\Delta\sigma}\ \frac{I(\sigma+\Delta\sigma)-I(\sigma)}{I(\sigma+\Delta\sigma)+I(\sigma)}\right] = \frac{1}{2}\frac{I'(\sigma)}{I(\sigma)}\ . \tag{18}$$

In other words, our original problem was reduced to the existence of solutions of the differential equation

$$\frac{I'(\sigma)}{I(\sigma)} = \Big[1-4\beta^2p^2\Big]\frac{\rho'}{\rho}\ + \Big[\frac{1}{1-\alpha^2p^2}\Big]\frac{\alpha'}{\alpha} - \Big[8\beta^2p^2\Big]\frac{\beta'}{\beta}\ . \tag{19}$$

ELASTIC IMPEDANCE

The *Elastic Impedance* function EI proposed by Connolly (1999) is obtained by equalling equation (15) to $\Delta EI/2EI$ (the discrete version of $EI'/2EI$) and applying difference calculus, with the additional assumption that θ and the ratio $K=\beta^2/\alpha^2$ are constant. The same result can be found directly from equation (19), resulting in the following differential equation for the elastic impedance function EI,

$$\frac{EI'}{EI} = \Big[1-4K\sin^2\theta\Big]\frac{\rho'}{\rho} + \Big[\sec^2\theta\Big]\frac{\alpha'}{\alpha} - \Big[8K\sin^2\theta\Big]\frac{\beta'}{\beta}\ . \tag{20}$$

The general solution for the above equation, under the mentioned assumptions, is given by

$$EI = EI_0\ \rho^{1\,-\,4K\sin^2\theta}\ \alpha^{\sec^2\theta}\ \beta^{-8K\sin^2\theta}\ , \tag{21}$$

where EI_0 is a normalization constant (see Whitcombe (2002)).

REFLECTION IMPEDANCE

We are interested on the existence of a general solution of equation (19), i.e., if there is a *Reflection Impedance* function RI, such that

$$\frac{RI'}{RI} = \Big[1-4\beta^2p^2\Big]\frac{\rho'}{\rho}\ + \Big[\frac{1}{1-\alpha^2p^2}\Big]\frac{\alpha'}{\alpha} - \Big[8\beta^2p^2\Big]\frac{\beta'}{\beta}\ , \tag{22}$$

for all possible choices of α, β and ρ. Clearly, the solution is not unique, since any multiple of it is also a solution.

As shown in Appendix A, equation (22) admits a closed-form solution only if β has a functional dependence on ρ, i.e., $\beta \equiv \beta(\rho)$. Under this assumption, the solution for RI is given by

$$RI = RI_0\ \frac{\rho\ \alpha}{\sqrt{1-\alpha^2p^2}}\ \exp\left\{-4p^2\left[\beta^2 + \int\frac{\beta^2}{\rho}d\rho\right]\right\}\ , \tag{23}$$

where RI_0 is a constant. A particularly simple formula is obtained by assuming a relationship of the form

$$\rho = b\ \beta^\gamma\ \text{, or equivalently, }\ \frac{\rho'}{\rho} = \gamma\ \frac{\beta'}{\beta}\ , \tag{24}$$

where b is some constant of proportionality and γ is a constant. In this case, solution (23) reduces to

$$RI = RI_0 \, \frac{\rho \, \alpha}{\sqrt{1-\alpha^2 p^2}} \cdot \begin{cases} \exp\{-2[2+\gamma]\beta^2 p^2\} & , \quad \beta' \neq 0 \\ \rho^{-4\beta^2 p^2} & , \quad \beta' = 0 \end{cases} \quad . \tag{25}$$

REDUCTION TO THE SIMPLE CASES

In the case of a normal incidence, both elastic (EI) and the reflection (RI) impedance functions reduce to a multiple of the acoustic impedance (AI), so the approximation for the reflection coefficient remains exact. Indeed,

$$\lim_{\theta \to 0} \frac{EI_2 - EI_1}{EI_2 + EI_1} = \lim_{p \to 0} \frac{RI_2 - RI_1}{RI_2 + RI_1} = \frac{AI_2 - AI_1}{AI_2 + AI_1} = R_0 \ . \tag{26}$$

However, for the case of non-normal incidence in acoustic media ($\beta = 0$), the elastic impedance approximation for R *does not reduce* to the exact one given by equation (7), as opposite to the reflection impedance approximation, where the exact expression is maintained. More explicitly,

$$\lim_{\beta \to 0} \frac{EI_2 - EI_1}{EI_2 + EI_1} = \frac{AI_2 \ \alpha_2^{\tan^2\theta} - AI_1 \ \alpha_1^{\tan^2\theta}}{AI_2 \ \alpha_2^{\tan^2\theta} + AI_1 \ \alpha_1^{\tan^2\theta}} \neq R_a \ , \tag{27}$$

and

$$\lim_{\beta \to 0} \frac{RI_2 - RI_1}{RI_2 + RI_1} = \frac{AI_2 \ \sec\theta_2 - AI_1 \ \sec\theta_1}{AI_2 \ \sec\theta_2 + AI_1 \ \sec\theta_1} = R_a \ . \tag{28}$$

APPLICATIONS

In this session, we use simple, but typical synthetic examples to examine the approximation for the P-P elastic reflection coefficient in terms of the elastic and reflection impedances. We discuss the approximations both for modelling and inversion purposes.

Modelling

In order to analyse the accuracy of EI and RI functions presented above, we consider a simple two-layer model in three different situations: weak, medium and large contrasts of the parameters. Table 1 summarizes the data.

We compare the exact reflection coefficient with its first-order approximation (see equation (15), as well as the impedance-type approximations of equation (10) under the use of the elastic impedance of equation (21) and reflection impedance of equation (25), respectively.

Model	Medium	α [km/s]	β [km/s]	ρ [g/cm^3]
Weak	Layer 1	3.20	1.50	2.30
	Layer 2	3.00	1.40	2.20
	Contrast	0.06	0.06	0.04
Medium	Layer 1	3.50	1.80	2.50
	Layer 2	3.00	1.40	2.20
	Contrast	0.15	0.25	0.13
Large	Layer 1	4.50	2.10	2.70
	Layer 2	3.00	1.40	2.20
	Contrast	0.40	0.40	0.20

Table 1: P- and S-wave velocities and densities for the numerical experiments.

For the elastic impedance approximation we have chosen, as usual in the literature,

$$K = \frac{(\beta_1/\alpha_1)^2 + (\beta_2/\alpha_2)^2}{2} \; . \tag{29}$$

Observe that in the large-constrast model the ratio β/α was made constant ($K = 0.218$) in order to offer the best conditions for the elastic impedance approximation. For the reflection impedance approximation (taking into account that $\beta_1 \neq \beta_2$) we set

$$\gamma = \frac{\ln(\rho_2/\rho_1)}{\ln(\beta_2/\beta_1)} \; . \tag{30}$$

The values for the constants EI_0 and RI_0 are irrelevant: any choice will produce the same value for the approximation of R.

For each situation we consider the complete range of reflection angles ($0 \leq \theta \leq 90^o$). This includes, of course, both pre- and post-critical reflections. The resulting approximations for the reflection coefficient are shown in Figures 1–3.

From the experiments, we conclude that the reflection impedance approximation has the best performance in all cases. In the case of post-critical reflections, the results are far better: all other approximations do not follow the correct shape of the exact curve. Therefore, there is a significant gain in accuracy provided by the reflection impedance approximation, as compared to the one that uses the elastic impedance.

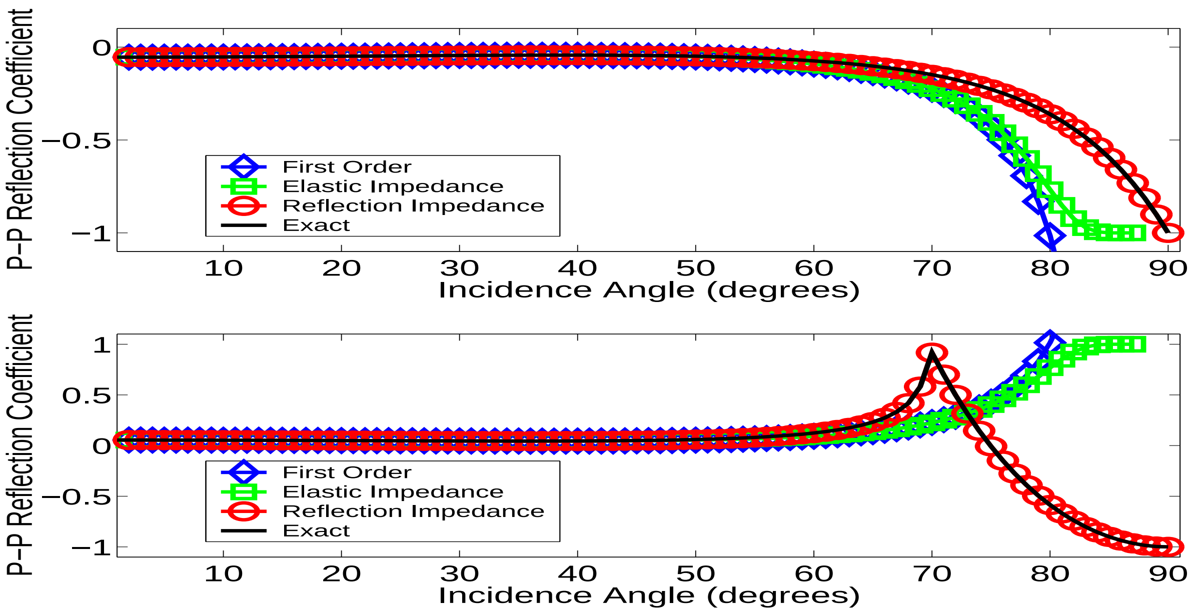

Figure 1: P-P reflection coefficient for the weak-contrast model given in Table 1: without (top) and with (bottom) post-critical reflections.

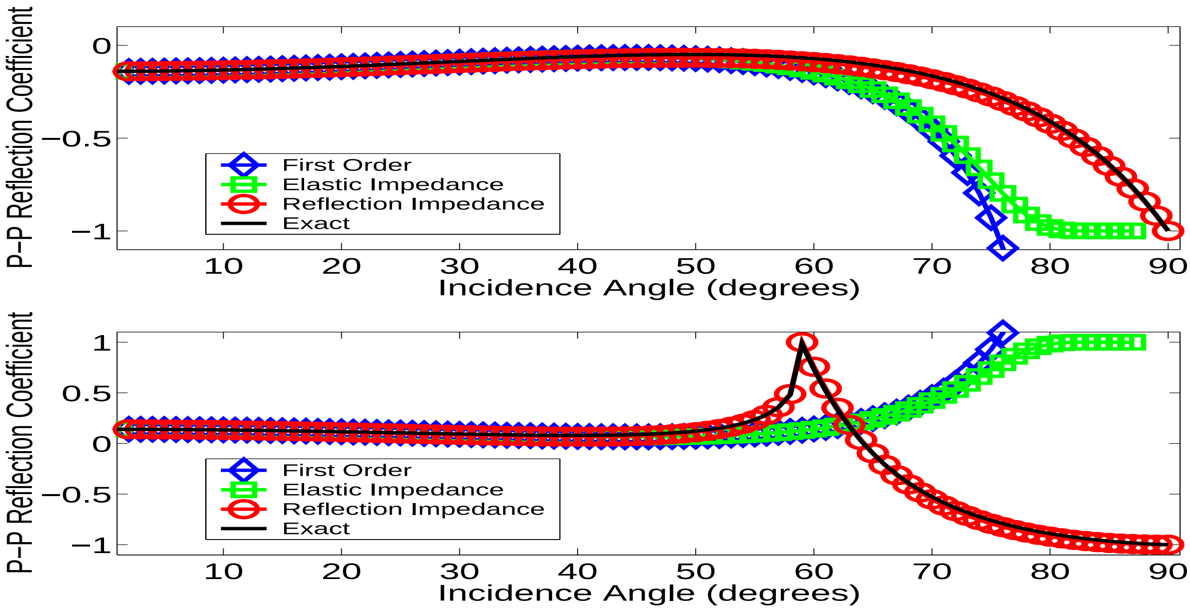

Figure 2: P-P reflection coefficient for the medium-contrast model given in Table 1: without (top) and with (bottom) post-critical reflections.

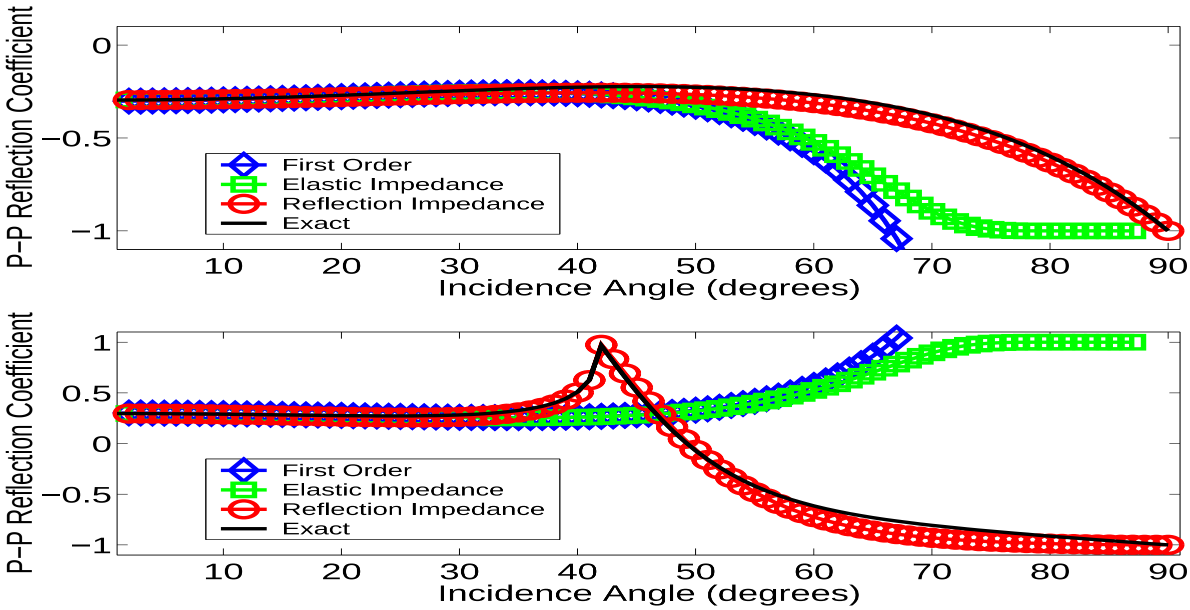

Figure 3: P-P reflection coefficient for the large-contrast model given in Table 1: without (top) and with (bottom) post-critical reflections.

AVO Inversion

We have also compared the performance of the three different approximations of R for the estimation of the intercept, A, and gradient, B, attributes, according to equation (16). The model parameters are the same as in the previous experiments. We have added a white noise of ratio 1:3 to the exact reflection coefficient's curve and then apply least-squares techniques to recover A and B. The details of the used numerical procedure are shown in Appendix B.

Tables 2 and 3 summarize the inversion results, where, again, we can observe that the inverted attributes using the reflection impedance approximation are of better accuracy than all the others. In Figures 4–6 we show the approximation for R using the inverted parameters and the corresponding approximation formulas.

Contrast	Reflection	Exact	Linear	EI	RI
Weak	Noncritical	– 0.0544	0.0360	0.0723	– 0.0345
	Critical	0.0544	0.1353	0.1111	0.0442
Medium	Noncritical	– 0.1401	– 0.0549	– 0.0398	– 0.1120
	Critical	0.1401	0.3141	0.2830	0.1372
Large	Noncritical	– 0.2960	– 0.2293	– 0.2099	– 0.2767
	Critical	0.2960	0.5724	0.5305	0.3318

Table 2: Results for the least-squares estimation of the Intercept parameter A.

Contrast	Reflection	Exact	Linear	EI	RI
Weak	Noncritical	0.0475	– 0.3266	– 0.4468	0.0559
	Critical	– 0.0475	– 0.3578	– 0.2563	– 0.0862
Medium	Noncritical	0.2273	– 0.2412	– 0.3125	0.1139
	Critical	– 0.2273	– 0.8563	– 0.6302	– 0.2952
Large	Noncritical	0.2373	– 0.1822	– 0.3084	0.1181
	Critical	– 0.2373	– 1.4034	– 0.8847	– 0.4690

Table 3: Results for the least-squares estimation of the Gradient parameter B.

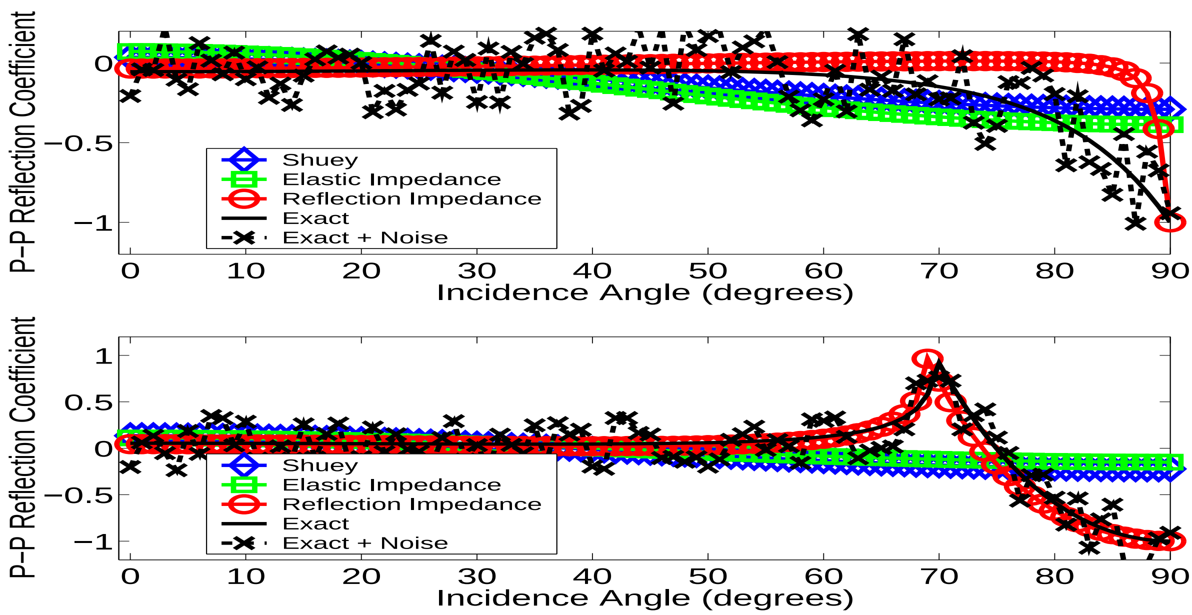

Figure 4: AVO curves for the weak-contrast model inverted parameters in Tables 2 and 3: without (top) and with (bottom) post-critical reflections.

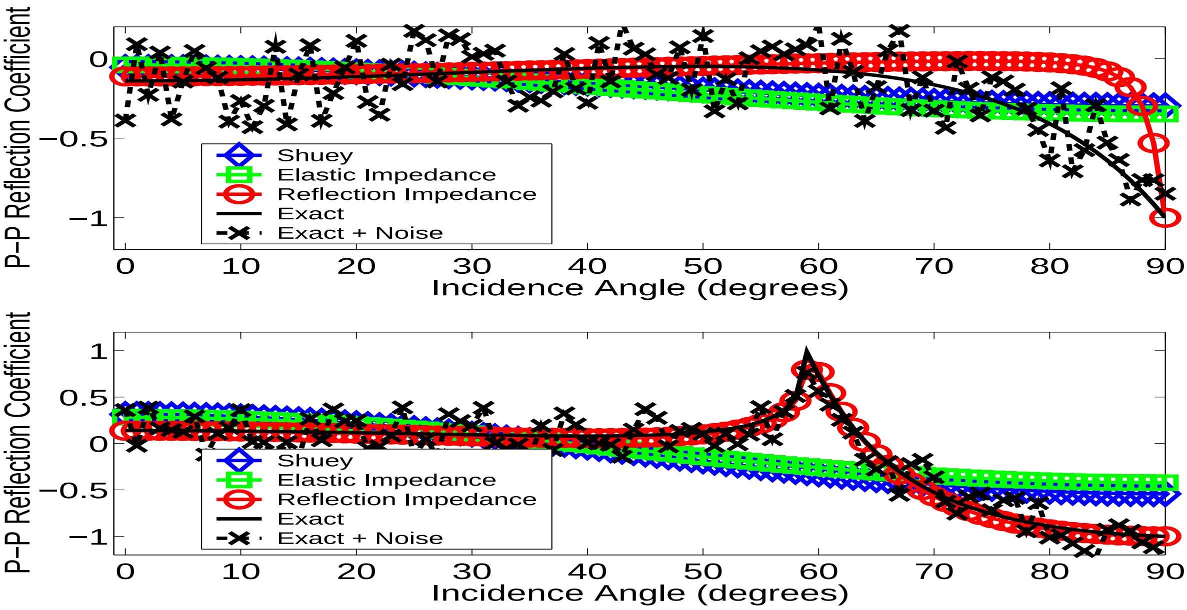

Figure 5: AVO curves for the medium-contrast model inverted parameters in Tables 2 and 3: without (top) and with (bottom) post-critical reflections.

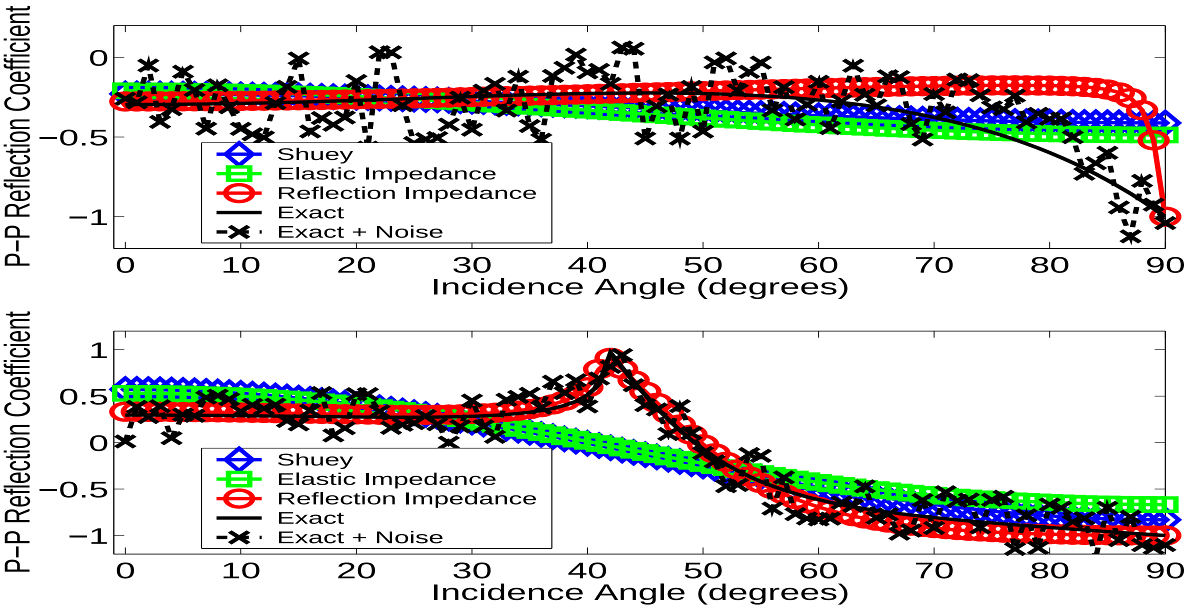

Figure 6: AVO curves for the large-contrast model inverted parameters in Tables 2 and 3: without (top) and with (bottom) post-critical reflections.

CONCLUSIONS

We have discussed the problem of determination and use of impedance functions generalizing the simple expression of the P-P reflection coefficient under normal incidence in acoustic/elastic media under oblique incidence in acoustic media, to oblique-incidence in elastic media. We have shown that for arbitrary selection of densities and P- and S-velocities, there is no closed-form impedance function fulfills the required task. Under additional, ad hoc, assumptions, impedance functions can be defined that provide useful approximations to the P-P reflection coefficients. We have examined two of such impedance functions available in the literature, namely, the elastic and reflection impedances, and discussed their potential for approximating the P-P reflection coefficient for modelling and inversion purposes. Our simple, but typical, numerical experiments have shown that the reflection impedance provide significantly better results, both for modelling and AVO inversion.

The elastic impedance has shown to provide good insight and results on calibration of seismic data for inversion purposes from well data (see Connolly (1999); Whitcombe (2002); Mallick (2001)). Current research is being done to employ a similar approach using, however the reflection impedance function. Furst results in this direction are shown in Santos et al. (2002).

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APPENDIX A

Let us assume that the differential equation (22) has a solution $RI \equiv RI(\rho, \alpha, \beta, \sigma)$ for any choice of the density and velocity functions. Under a vertically inhomogeneous assumption, as previously indicated, the rayparameter, p, has a constant value (that is, it does not depend on the medium parameters and also does not depend on σ). Therefore, the total differential for the function RI is

$$RI' = \frac{\partial RI}{\partial \rho}\rho' + \frac{\partial RI}{\partial \alpha}\alpha' + \frac{\partial RI}{\partial \beta}\beta' + \frac{\partial RI}{\partial \sigma} \ , \tag{31}$$

Hence, to satisfy equation (22), we must have

$$\frac{1}{RI}\frac{\partial RI}{\partial \rho} = \frac{1-4\beta^2p^2}{\rho} \ , \quad \frac{1}{RI}\frac{\partial RI}{\partial \alpha} = \frac{1}{\alpha(1-\alpha^2p^2)} \ , \quad \frac{1}{RI}\frac{\partial RI}{\partial \beta} = -8\beta p^2 \ \text{and} \ \frac{1}{RI}\frac{\partial RI}{\partial \sigma} = 0 \ . \tag{32}$$

From the last condition, we conclude that RI does not depend on σ. Using the condition for the β-term, it follows that RI has the form

$$RI = G(\rho, \alpha)\exp\{-4\beta^2p^2\} \ , \tag{33}$$

where G is some function to be determined. Substituting the above expression in the ρ-term in equation (32) we arrive at

$$\frac{1}{G}\frac{\partial G}{\partial \rho} = \frac{1-4\beta^2p^2}{\rho} \ , \tag{34}$$

which is impossible since G does not depend on β. This shows that RI cannot have a closed-form expression that is valid for all medium parameters, ρ, α and β.

This can be overcome, for example, by considering that β has a functional dependence on ρ, i.e., $\beta \equiv \beta(\rho)$. With such an assumption, relations (32) turn out to be

$$\frac{1}{G}\frac{\partial G}{\partial \rho} = \frac{1-4\beta^2p^2}{\rho} \ , \ \text{and} \quad \frac{1}{G}\frac{\partial G}{\partial \alpha} = \frac{1}{\alpha(1-\alpha^2p^2)} \ . \tag{35}$$

The differential equation for α is easily solved giving

$$G = H(\rho)\frac{\alpha}{\sqrt{1-\alpha^2 p^2}} \ , \tag{36}$$

with H being some function. Substituting the above solution into the differential equation in ρ yields

$$\frac{1}{H}\frac{\partial H}{\partial \rho} = \frac{1-4\beta^2 p^2}{\rho} \tag{37}$$

and so,

$$H = RI_0 \ \rho \ \exp\left\{-4p^2 \int \frac{\beta^2}{\rho} d\rho\right\} \ , \tag{38}$$

where RI_0 is a constant. Collecting results, we finally conclude that

$$RI = RI_0 \ \frac{\rho \ \alpha}{\sqrt{1-\alpha^2 p^2}} \ \exp\left\{-4p^2\left[\beta^2 + \int \frac{\beta^2}{\rho} d\rho\right]\right\} \ , \tag{39}$$

APPENDIX B

The P–wave reflection coefficient can be approximated by Shuey's two-term approximation (Shuey (1985)),

$$R \approx A + B\sin^2\theta \ , \tag{40}$$

where A is the *AVO intercept*, namely the normal incidence P-wave reflection coefficient,

$$A = R_0 \approx \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\right] \ , \tag{41}$$

and B is the AVO gradient (or slope),

$$B = \frac{1}{2}\frac{\Delta\alpha}{\alpha} - 2\frac{\beta^2}{\alpha^2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right] \ . \tag{42}$$

Using simple least-squares procedures, the AVO indicators A and B can be directly estimated from amplitudes *versus* angle of incidence data. The expression of the reflection coefficient in terms of elastic (EI) and reflection (RI) impedances can also be used to estimate the same indicators. The procedure is described as follows.

First, from the impedance concept, formula (10), we define a new quantity F as the ratio of the impedances, $F = I_2/I_1$. Therefore,

$$R = \frac{I_2 - I_1}{I_2 + I_1} = \frac{I_2/I_1 - 1}{I_2/I_1 + 1} = \frac{F-1}{F+1} \ , \tag{43}$$

and

$$F = \frac{1+R}{1-R} \ . \tag{44}$$

From the elastic impedance function (21), and using the same approximation to obtain (40), i.e., $\tan^2\theta \approx \sin^2\theta$, we arrive at

$$\ln F = \ln\left[\frac{EI_2}{EI_1}\right] = \ln\left[\frac{\rho_2\alpha_2}{\rho_1\alpha_1}\right] + \ln\left[\frac{\alpha_2}{\alpha_1}\left(\frac{\rho_2\beta_2^2}{\rho_1\beta_1^2}\right)^{-4K}\right]\sin^2\theta = \Lambda_1 + \Lambda_2\sin^2\theta \ . \tag{45}$$

After a simple application of a linear least-squares approximation to obtain the best Λ_1 and Λ_2 that fits $\ln F$ and $\sin^2\theta$, the AVO indicators A and B are given by

$$A = \frac{\exp\{\Lambda_1\} - 1}{\exp\{\Lambda_1\} + 1} \ , \text{ and } B = \frac{\exp\{\Lambda_2\} - 1}{\exp\{\Lambda_2\} + 1} \ . \tag{46}$$

Now, from the reflection impedance function (25), and assuming $\beta_1 \neq \beta_2$, we have

$$F = \frac{RI_2}{RI_1} = \frac{\rho_2\alpha_2}{\rho_1\alpha_1}\frac{\cos\theta}{\sqrt{1-\alpha_2^2p^2}}\exp\{-2[2+\gamma][\beta_2^2-\beta_1^2]p^2\} = \Lambda_1\frac{\cos\theta}{\sqrt{1-\Lambda_2^2\sin^2\theta}}\exp\{\Lambda_3\sin^2\theta\}\ . \quad (47)$$

Again, we can find Λ_1, Λ_2 and Λ_3 in a least-squares sense. It is important to note that here it is not possible to "linearize" the expression to apply linear least-squares: a nonlinear solver must be used. Nevertheless, since there are only three unknowns, the procedure is easily implemented. After some algebraic manipulations, the AVO indicators A and B can be recast as functions of the Λ's

$$A = \frac{\Lambda_1 - 1}{\Lambda_1 + 1}\ , \text{ and } B = \frac{\Lambda_2 - 1}{\Lambda_2 + 1} + \frac{\Lambda_3}{2}\ . \quad (48)$$