2.5-D finite-difference solution of the acoustic wave equation

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ABSTRACT

Finite differences applied to the full 3-D wave equation is a rather time consuming process. However, in the 2.5-D situation, we can take advantage of the medium symmetry. By taking the Fourier transform in the out-of-plane direction, the 3-D problem can be reduced to a sequence of 2-D ones. The third dimension is taken in to account by a sum over the corresponding wave vector component. In this way, the 2.5-D finite differences program can be improved by a factor that increases with the size of the model. Even for relatively small models, this procedure reduces the computation time by a factor of about ten. The modeling results obtained by the new 2.5-D finite-difference scheme are of comparable quality to a standard 3-D finite-difference scheme.

INTRODUCTION

Finite difference modeling of wave propagation in heterogeneous media is a useful technique in a number of disciplines, including seismology and ocean acoustics, among others. However, the size of the models that can be treated by finite difference methods in three spatial dimensions has been rather limited except on supercomputers.

In other forward modeling schemes, the medium symmetry in the so-called 2.5-D situation has been made use of in order to reduce the computational costs (see, e.g., Novais, 1998). The attribute 2.5-D designates a situation where the medium depends on two spatial coordinates only and the seismic line is parallel to the symmetry axis.

In this work, we show how a finite difference scheme can be adapted to the 2.5-D situation. The full 3-D finite-difference scheme can be reduced to a repeated 2-D FD scheme by applying the Fourier transform with respect to the out-of-plane coordinate to the 3-D wave equation and using the medium symmetry. The resulting 2-D equations are solved by finite differences for different values of the wave vector component. The necessary inverse Fourier transform is realized by a simple sum over all 2-D finite-difference results in order to obtain the full 3-D wavefield.

The technique was originally proposed by Zhou & Greenhalgh (1998), using a finite element method in the frequency domain. Here we present the same approach for a finite-difference scheme applied in the time domain.

2.5-D SOLUTION

We study seismic wave propagation governed by the acoustic wave equation

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{v^2} u_{tt} - f(t)\delta(x - x_s)\delta(y - y_s)\delta(z - z_s) \ , \tag{1}$$

where $u = u(x, y, z, t)$ is the acoustic wavefield, $v = v(x, y, z)$ is the velocity of the medium under consideration, and $f(t)$ is a band-limited source function located at (x_s, y_s, z_s).

We assume that the velocity wavefield is a function of x and z only, i.e., $v = v(x, z)$ and the source is located in the symmetry plane ($y_s = 0$). Applying the Fourier transform in the out-of-plane y-coordinate, the 3-D wave equation (1) can be reduced to the following form

$$U_{xx} - \kappa^2 U + U_{zz} = \frac{1}{v^2} U_{tt} - f(t)\delta(x - x_s)\delta(z - z_s) \ , \tag{2}$$

where

$$U(x, \kappa, z, t) = \int_{-\infty}^{\infty} dy \ u(x, y, z, t) \ e^{-i\kappa y} \ . \tag{3}$$

The solution of equation (2) will be obtained by a finite-difference scheme followed by the application of the inverse Fourier transform

$$u(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa \ U(x, \kappa, z, t) \ e^{i\kappa y} \ . \tag{4}$$

In particular, in the symmetry plane, i.e., for $y = 0$ we have

$$\begin{aligned} u(x, 0, z, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa \ U(x, \kappa, z, t) \\ &\approx \frac{\Delta\kappa}{\pi} \sum_{k_n \geq 0} U(x, \kappa_n, z, t) \ , \end{aligned} \tag{5}$$

where $\Delta\kappa$ is the uniform increment in κ and we have used the fact that $U(x, \kappa, z, t)$ is an even function in κ. Equation (5) means that the field $u(x, 0, z, t)$ can be obtained by summing the contributions for all $\kappa_n \geq 0$.

FINITE-DIFFERENCE FORMULAS

A set of indices i, j and l is chosen to establish a finite-difference scheme with uniform grid spacing Δx, Δz and Δt in x, z and t, respectively: $x_i = x_{min} + i \ \Delta x$, $z_j = z_{min} + j \ \Delta z$ and $t_l = t_{min} + l \ \Delta t$. Consequently, we denote, for a fixed κ, $U(x_i, \kappa, z_j, t_l) = U^l_{i,j}$.

The finite-difference scheme for solving equation (2) was chosen to be fourth-order in space and second-order in time, and is given by

$$\begin{aligned} U_{i,j}^{l+1} = -\frac{1}{12} \quad \{ \quad & a_{i,j}\left[U_{i-2,j}^{l} + U_{i+2,j}^{l} - 16\left(U_{i-1,j}^{l} + U_{i+1,j}^{l}\right) + 30U_{i,j}^{l}\right] \\ + \quad & b_{i,j}\left[U_{i,j-2}^{l} + U_{i,j+2}^{l} - 16\left(U_{i,j-1}^{l} + U_{i,j+1}^{l}\right) + 30U_{i,j}^{l}\right]\} \\ + \quad & \Delta t^2\kappa^2 U_{i,j}^{l} + 2U_{i,j}^{l} - U_{i,j}^{l-1} + f_{i,j}^{l}\ , \end{aligned} \tag{6}$$

where

$$a_{i,j} = (v_{i,j}\Delta t/\Delta x)^2,\ b_{i,j} = (v_{i,j}\Delta t/\Delta z)^2, \tag{7}$$

$v_{i,j}$ denotes the velocity at (x_i, z_j), and

$$f_{i,j}^{l} = \begin{cases} f(t_l)\ , & \text{if } x_i = x_s \text{ and } z_j = z_s, \\ 0\ , & \text{otherwise.} \end{cases} \tag{8}$$

For initiating the propagation process, we set

$$U_{i,j}^{0} = 0,\ \text{for all } i, j\ , \tag{9}$$

and define the boundary conditions

$$U_{i,0}^{l} = U_{0,j}^{l} = 0,\ \text{for all } i, j, l\ . \tag{10}$$

We assume a uniform grid spacing such that $\Delta x = \Delta z = h$. Then, the maximum value for the grid spacing h that can be used without causing excessive dispersion of energy, is determined by the condition (Mufti, 1990)

$$h \leq \frac{v_{min}}{\vartheta f_{max}}. \tag{11}$$

Here, v_{min} is the minimum value of the velocity field, f_{max} is the maximum frequency of the source pulse, and ϑ is the number of samples per minimum wavelength (to be chosen). Moreover, for a given value of the grid spacing, the process becomes numerically unstable unless the time sampling interval satisfies the condition

$$\Delta t \leq \frac{\gamma\ h}{v_{max}}\ , \tag{12}$$

where v_{max} represents the maximum value of velocity field and γ is a constant. According to Mufti et al. (1996), the optimal values for the above parameters are $\gamma = 0.5$ and $\vartheta = 3.5$.

NUMERICAL EXPERIMENTS

We illustrate the 2.5-D finite-difference process discussed above by means of two simple synthetic experiments. The first model consists of a planar interface (Figure 1), separating two homogeneous halfspaces with velocities 3.0 km/s and 3.5 km/s. The density in both halfspaces

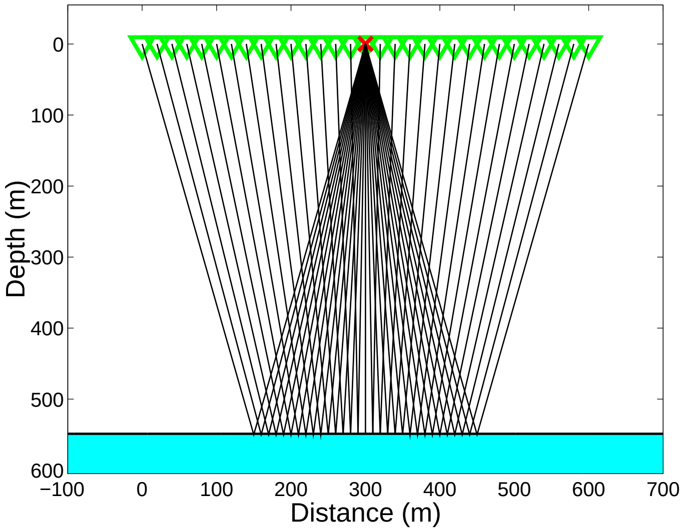

Figure 1: First model and ray family for the common-shot experiment.

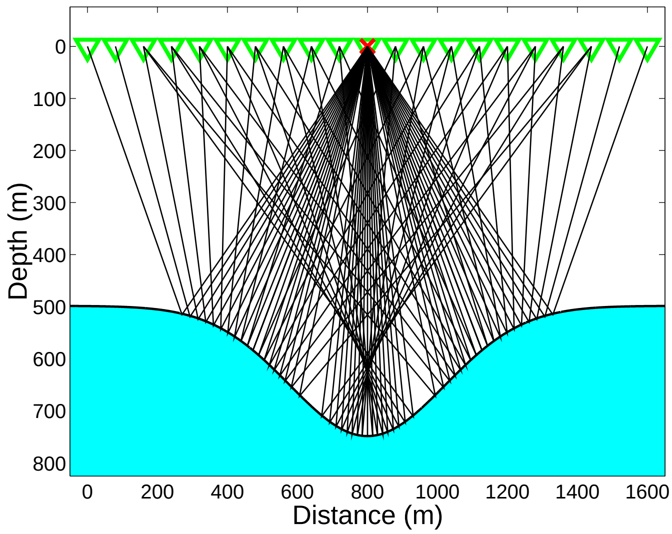

Figure 2: Second model and ray family for the common-shot experiment.

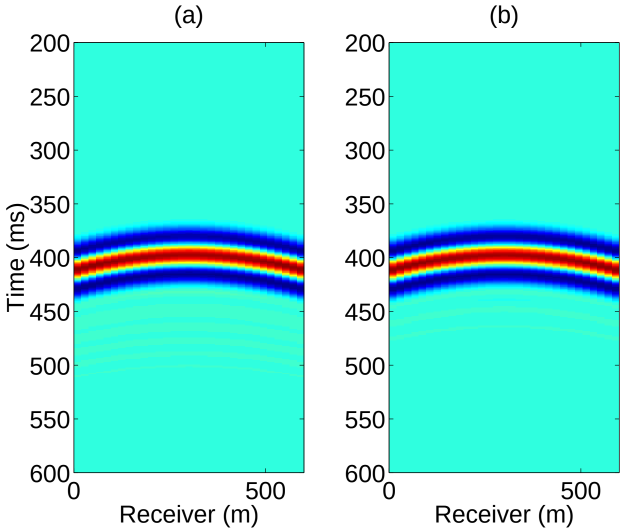

Figure 3: First model - synthetic seismograms: (a) 2.5-D Finite-differences, (b) 3-D Finite-differences.

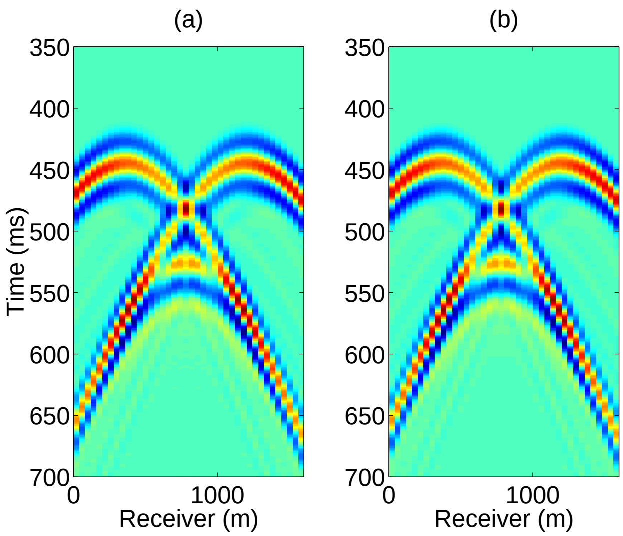

Figure 4: Second model - synthetic seismograms: (a) 2.5-D Finite-differences, (b) 3-D Finite-differences.

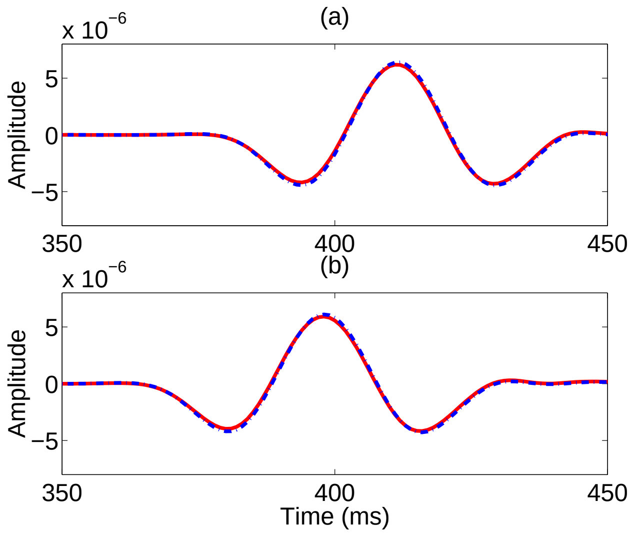

Figure 5: First model: (a) Comparison of traces modeled by 2.5-D FD (solid red line) and 3-D FD (dash-dotted blue line). (a) Trace at $x = 0$m. (b) Trace at $x = 300$m.

is constant and equal to unity. For this model, we have simulated a split-spread experiment with a omnidirectional point source located at $x = 300$ m and 31 receivers equally spaced at every 20 m between $x = 0$ m and $x = 600$ m.

The second model is an anticlinal interface (Figure 2), again separating two homogeneous halfspaces with velocities 3.0 km/s and 3.5 km/s. As before, the density in both halfspaces is constant and equal to unity. For this model, we have simulated a split-spread experiment with a omnidirectional point source located at $x = 800$ m and 41 receivers equally spaced at every 40 m between $x = 0$ m and $x = 1600$ m. In this situation, the recorded wavefield has encountered a caustic.

For both models, the simulated seismic common-shot sections resulting from the 2.5-D finite-difference scheme have been compared with the corresponding sections obtained using 3-D finite differences (First model: Figure 3, and Second model: Figure 4). In both methods we have used a uniform spatial grid with $\Delta x = \Delta y = \Delta z = 10$ m. The time sampling interval is $\Delta t = 1$ ms. As the source wavelet we have chosen a Küpper wavelet with f_{max} =35 Hz. The summation to realize the inverse Fourier transform was performed with $\Delta\kappa = 1/1200$ m^{-1} and $\kappa_{max} = 0.1$ m^{-1}.

A trace-by-trace comparison reveals almost no differences between the two modeling results (Figure 5 and Figure 6).

For a more quantitative analysis, the peak amplitudes along the reflections have been picked for all seismograms. These are show in Figure 7 (a), first model, and Figure 8 (a), second model. The 2.5-D finite-differences overestimate slightly the amplitudes as compared to 3-D finite-differences.

Figures 7(b) and 8(b) show the relative error between 2.5-D finite-difference and 3-D finite-

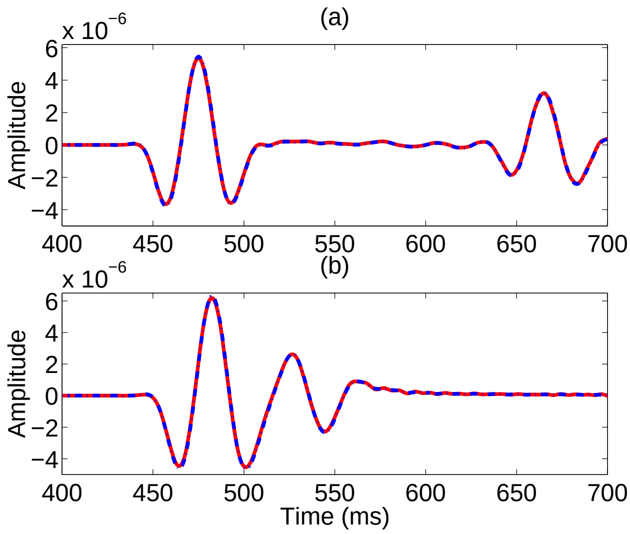

Figure 6: Second model: (a) Comparison of traces modeled by 2.5-D FD (solid red line) and 3-D FD (dash-dotted blue line). (a) Trace at $x = 0$m. (b) Trace at $x = 800$m.

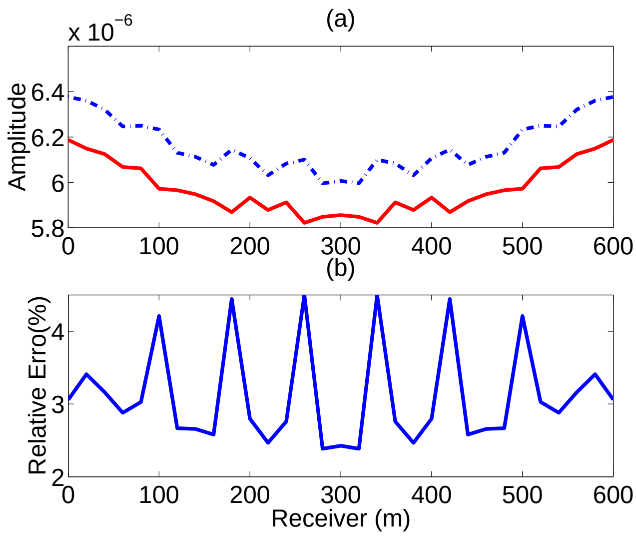

Figure 7: First model: (a) Comparison of the peak amplitudes: 2.5-D FD (dashed-dotted blue line) 3-D FD (solid red line), (b) Relative error.

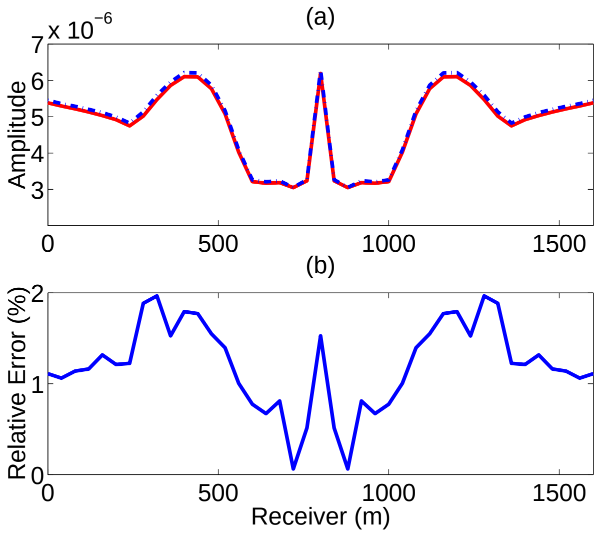

Figure 8: Second model: (a) Comparison of the peak amplitudes: 2.5-D FD (dashed-dotted blue line) 3-D FD (solid red line), (b) Relative error.

difference. Observe that the relative error in both models is less then 5%.

CONCLUSIONS

We have used the approach of Zhou and Greenhalgh (1998) with an application to finite differences in the time domain. For two simple models of smooth reflectors between two homogeneous acoustic media, we have computed the seismogram using the described 2.5-D finite-difference scheme. For the same models, we have also computed, as a reference, the corresponding seismogram using a 3-D finite-difference scheme. We observed in both cases that the wavefield modeled by 2.5-D finite differences agrees very well with the 3-D result.

It came as a surprise to us that the coincidence between the 2.5-D and 3-D wavefields is even better in the second model that involves a caustic than in the first one that consists only of a horizontal planar reflector. The relative amplitude error for the horizontal-reflector model is between two and five percent, whereas for the caustic model it never exceeds two percent.

NEXT STEP

A presently open question regards the sampling of the wavenumber in the out-of-plane direction. To be more specific, we are going to investigate what sampling interval and what maximum wavenumber are necessary and sufficient to obtain an acceptable 2.5-D modeling result.

Further studies on the subject will also include investigations on the method in more realistic earth models.

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