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An FD eikonal solver for 3-D anisotropic media

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ABSTRACT

To compute the traveltime of seismic waves in a general anisotropic medium it is possible to use a perturbation approach which is based on approximating this medium by a simpler reference medium. An elliptically anisotropic medium approximates a strong anisotropic medium better than an isotropic one does and the corresponding eikonal equation is only slightly more complex then for the isotropic case. We consider an elliptically anisotropic medium as a reference medium and a strong anisotropic medium as a perturbed one for the perturbation method. Traveltimes in the reference medium are computed with an FD eikonal solver which allows fast and highly accurate calculation of traveltimes for elliptically anisotropic media. To achieve stability a wave front expansion scheme is applied.

INTRODUCTION

Anisotropy has been recognized as an important feature of seismic wave propagation. There is an interest in extending methods of exploration seismology to anisotropic media. The computation of traveltimes in anisotropic media is expensive because for each propagation step an eigenvalue problem must be solved. Therefore we want to use a perturbation technique which is based on approximating an anisotropic medium by a simpler, analytically treatable, reference medium. Differences between both media are taken into account by adding corrections to the traveltime obtained for the reference medium.

We consider an elliptically anisotropic medium as a reference medium. In elliptical anisotropy the eikonal equation for the considered wave type is only slightly more complex compared to the isotropic case. Elliptical anisotropy is of limited practical significance, since the media of elliptical symmetry hardly exist. The FD code for elliptically anisotropic media can be understood as a basic routine for computation of traveltimes in arbitrary anisotropic media. If the deviation between an elliptically anisotropic model and the model with given anisotropy allows for a linearization of

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traveltimes, a computation by a first order perturbation method embedded into the FD scheme is possible.

Perturbation method embedded into an FD scheme for weak anisotropy was considered by (Ettrich, 1998) for the 3-D case and by (Ettrich and Gajewski, 1998) for the 2-D case. For application to anisotropic media an elliptically anisotropic reference model gives higher accuracy but using an isotropic reference model gives higher speed of calculation.

To minimize errors which are inherent in the perturbation approach one should choose the reference medium as close as possible to the given anisotropic medium. In the paper by (Ettrich et al., 2000) the problem for the approximation of an arbitrary anisotropic medium by an ellipsoidal medium were derived. This approximation works well for P-anisotropies of up to 10 % when the polarization vector is substituted by the phase normal.

We want to apply the FD perturbation method to strong anisotropy. Following (Burridge et al., 1993) there are three possibilities to simplify general anisotropy to ellipsoidal symmetry and we use these possibilities for the construction of a reference ellipsoidal medium. Taken the physics of wave propagation into account we minimize the average of differences between the Christoffel matrices of the anisotropic medium and the ellipsoidal medium. This paper presents first results in this approach.

FD APPROXIMATION OF THE EIKONAL EQUATION

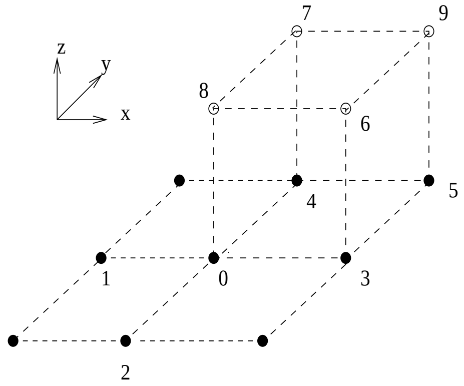
If one type of wave (quasi P-wave, quasi S_1- or S_2-wave) in an elliptically anisotropic medium is defined by three phase velocities (v_1, v_2, v_3) along each axis in the crystal coordinate system and by three angles (θ, ϕ, ξ) describing the orientation of the crystal coordinate system with respect to the global Cartesian coordinate system, the slowness surface (eikonal equation) for this wave reads:

$$(\mathbf{p}, R\,\mathbf{p}) = 1, \tag{1}$$

where $\mathbf{p}$ is the slowness vector, R is a 3×3 symmetric matrix, $(\,)$ denotes scalar product. In the crystal coordinate system this matrix is a diagonal matrix and reads $\tilde{R} = v_i^{-2}\delta_{ij}$ where δ_{ij} is the Kronecker-delta. In the global Cartesian coordinate system R is a function of v_1, v_2, v_3 and θ, ϕ, ξ. R and $\tilde{R}$ are connected by $R = U^T\,\tilde{R}\,U$, where U is a rotation matrix which defines the orientation of the crystal coordinate system with respect to the global Cartesian coordinate system.

The eikonal equation allows for the computation of one slowness vector component if the others are known and for the computation of the phase velocity for a given phase direction. In our algorithm the approximation of the eikonal equation from (Ettrich, 1998) is used. If there are traveltimes in gridpoints 0, 3, 4, 5, 6, 7 and 8 (see Fig. 1)

Figure 1: Scheme of expansion. The timed points are given by the black circle. The computation starts at point 0 with the minimum traveltime. Using traveltimes in points 0, 1, 2, 3 and 4 the traveltime in 8 can be calculated (scheme 3). Using traveltimes in points 0, 2, 3, 4 and 8 the traveltime in 6 can be calculated (scheme 2). To find the traveltime in points 9 the traveltimes in points 0, 3, 4, 5, 6 and 7 are used (scheme 1).



the ansatz used by (Ettrich, 1998) for the searched traveltime t_9 at the corner point of a cubic cell is:

$$
\begin{aligned}
(t_9 - t_0) = w_0 + w_1[(t_5 - t3)^2 + (t_7 - t_8)^2] + w_2[(t_5 - t_4)^2 + (t_6 - t_8)^2] \quad + \\
w_3[(t_6 - t_3)^2 + (t_7 - t_4)^2] + w_4[(t_3 - t_4)^2 + (t_6 - t_7)^2] \quad + \\
w_5[(t_6 - t_5)^2 + (t_8 - t_4)^2] + w_6[(t_7 - t_5)^2 + (t_8 - t_3)^2],
\end{aligned}
\tag{2}
$$

where:

$$
\begin{aligned}
w_1 &= 1.5 - \frac{w_0}{4h^2}(2v_4^2 + 2v_6^2 - v_1^2 - v_3^2), \\
w_2 &= 1.5 - \frac{w_0}{4h^2}(2v_4^2 + 2v_5^2 - v_2^2 - v_3^2), \\
w_3 &= 1.5 - \frac{w_0}{4h^2}(2v_5^2 + 2v_6^2 - v_1^2 - v_2^2), \\
w_4 &= \frac{w_0}{4h^2}(2v_4^2 - v_1^2 - v_2^2) - 0.5, \\
w_5 &= \frac{w_0}{4h^2}(2v_6^2 - v_2^2 - v_3^2) - 0.5, \\
w_6 &= \frac{w_0}{4h^2}(2v_5^2 - v_1^2 - v_3^2) - 0.5, \\
w_0 &= \frac{3h^2}{v_7^2}
\end{aligned}
\tag{3}
$$

and t_i is traveltime in gridpoints with number i and point 0 is the gridpoints with minimum traveltime, v_i, $i = 1, \ldots, 6$ are the velocities in X-, Y- and Z-direction (directions 1, 2, 3) and in direction of the diagonals $X - Z$, $X - Z$ and $Y - Z$ (directions 3, 4, 5). Formula (2) with coefficients (3) is analogous formula 1 from (Vidale, 1990), called scheme 1, applied to the majority of grid points.

Other two approximations are similar to formulas 3 and 4 from (Vidale, 1990). If the traveltime is known at gridpoints 0, 1, 2, 3, 4 and the traveltime in gridpoint 0 is minimum (see Fig. 1), to compute traveltime t_8 slowness vector components p_x and p_y are approximated by $p_x = (t_3 - t_1)/(2h)$ and $p_y = (t_4 - t_2)/(2h)$, and the eikonal equation (1) is solved for the remaining components $p_z = (t_6 - t_0)$ (scheme 3). If the traveltimes are known in gridpoints 0, 2, 3, 4 and 8 and the traveltime in gridpoint 0 is minimum (see Fig. 1), the traveltime t_6 is obtained by approximating $p_y = (t_4 - t2)/(2h)$, $p_x = (t_3 - t_0 + t_6 - t_8)/(2h)$ and $p_z = (t_8 - t_0 + t_6 - t_3)/(2h)$. With these expressions for the slowness vector components the eikonal equation (1) is solved for the searched traveltime t_6 (scheme 2). Detailed descriptions of the scheme were published by (Ettrich, 1998).

Scheme of expansion

To retain causality and to guarantee stability we expand wavefronts (Qin et al. 1992). The propagation of energy and, therefore, the causal continuation of computation is governed by the group velocity vectors. In isotropic models where group and phase velocity vectors coincide and the causality is achieved by sorting the outer points of the irregular volume of timed points with respect to traveltime from minimum to maximum. In anisotropic media the group velocity vector and the phase velocity vector does not coincide. The group velocity in elliptically anisotropy is given by the simple formula: $\mathbf{v}_{gr} = R\,\mathbf{p}$. Therefore it is not sufficient to carry out the calculation from point with minimum traveltime to point with maximum traveltime. For every eikonal solution we must compare the direction of the group velocity with the direction of the scheme of expansion. If the directions coincide the step is done. Otherwise, that point is not a point for a casual expansion and it is necessary to go to the next point. After every successful step the outer points of the timed points are sorted again with respect to traveltime from minimum to maximum.

Numerical results

We give numerical results for two models. Using a homogeneous elliptically anisotropic model we can check the accuracy of the described FD scheme. The phase velocities in the crystal coordinate system are 2 km/s along the X-axis, 2.4 km/s along the Y-axis and 2.8 km/s along the Z-axis. The crystal coordinate system is rotated by 30^o consecutively around the X, Y and Z axes. The grid spacing is 20 m. A cubic region of seven grid points around the source is initialized using the formula $t = \sqrt{\mathbf{x},\, \mathbf{R}^{-1}\mathbf{x}}$, where $\mathbf{x}$ is the radius-vector from the source to the gridpoint. Fig. 2 displays numerically and analytically computed wavefronts (left) and relative errors of the computation (right). The maximum relative error does not exceed 0.4 %.

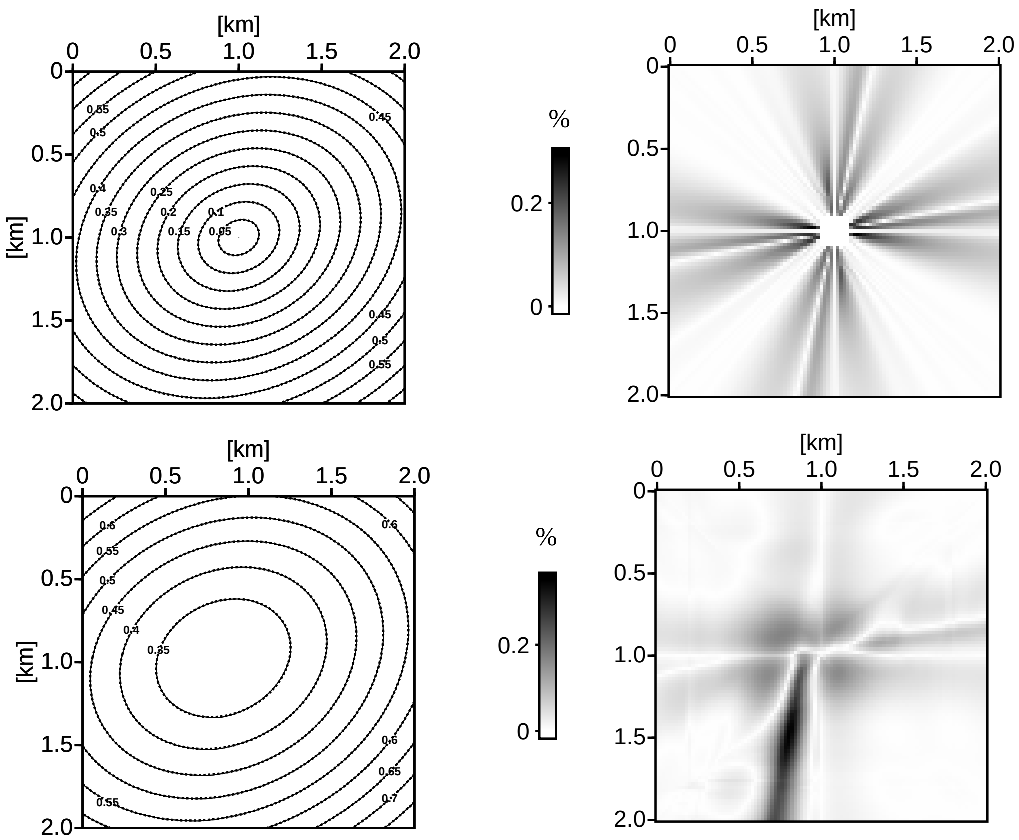


Figure 2: Wavefronts in a homogeneous elliptically anisotropic model: $X - Z$-slices with offset 0 km (upper, left) and 0.8 km (lower, left) from the source. The numerical solution is shown by solid lines, the analytical one is shown by dotted lines. On the right relative errors corresponding to the pictures on the left are given.

The second example is a horizontally layered model. The parameters of the upper layer are equal to the parameters of the homogeneous model considered above. At a depth of z = 0.6 km the phase velocities increase by 1 km/s and the crystal coordinate system is rotated by additional 20^o around each axis. A third layer at a depth of 1 km has velocities 4.6 km/s, 3.8 km/s and 4.8 km/s along X, Y and Z respectively. Angles are increased by 10^o. The model was smoothed. The wavefronts in Fig. 3 demonstrate the applicability of the method for such models with strong velocity contrasts.

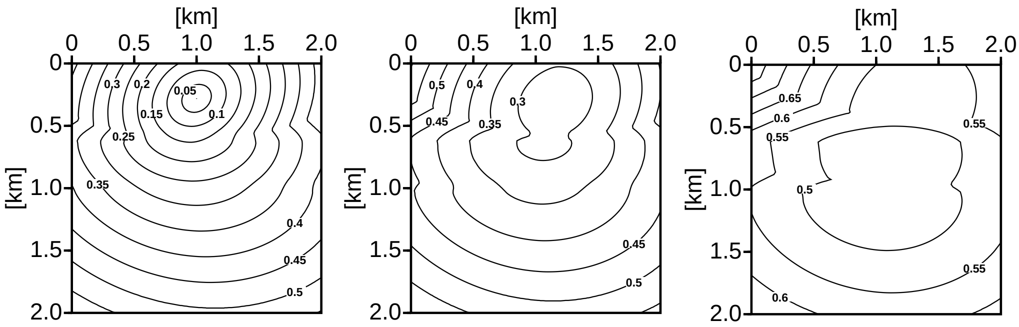


Figure 3: Wavefronts in a horizontally layered model; $X - Z$-slices with offset 0 km (left), 1 km (middle) and 1.3 km from the source.

TRAVELTIME PERTURBATION

To compute traveltimes for arbitrary anisotropic media, a perturbation scheme can be embedded into the FD eikonal solver. We consider a model with arbitrary anisotropy as a perturbed model with respect to an elliptically anisotropic reference model. The formula for traveltime correction derived by Cervený (1982) is used:

$$\Delta t(P,S) = -\frac{1}{2}\int_{t(S)}^{t(P)} \Delta\lambda_{ijkl} p_i p_l A_j A_k dt,$$

where:

$$\Delta\lambda_{ijkll} = \lambda_{ijkll} - \lambda_{ijkll}^{(0)}$$

with λ_{ijl} as density normalized elastic coefficients of the anisotropic medium, p_i are the components of the slowness vector, and A_j are the components of the polarization vector of the considered type of wave (qP, qS_1 or qS_2). The vectors $\mathbf{p}$ and $\mathbf{A}$ depend on to the reference ellipsoidal medium.

Theoretically these corrections can be of arbitrary order. However, for practical applications in most cases a first-order correction as above is used. To minimize the errors in the perturbation approach the reference medium should be chosen close to the given anisotropic medium. Formulas for a best-fitting ellipsoidal reference medium were derived by (Ettrich et al., 2000). In these formulas it is supposed that the polarization vector is substituted by the phase normal. Therefore only for weak anisotropy the phase velocity of the P-wave is well approximated.

We now consider the case of strong anisotropy. Following (Burridge et al., 1993) there are three possibilities to simplify orthorhombic or transversely isotropic symmetry to ellipsoidal one. The othorhombically anisotropic medium has an elliptical symmetry if non-zero elastic moduli satisfy:

$$
\begin{array}{lll}
A) & c_{44} = -c_{23}, \quad c_{55} = -c_{13}, \quad c_{66} = -c_{12}; & \\
B) & c_{44} = c_{55} = -c_{13} = -c_{23} \qquad c_{66} = (c_{11} + c_{22} + 2c_{12})^{-1}(c_{11}c_{22} - c_{12}^2); & (4) \\
C) & c_{44} = c_{55} = c_{66} = (c_{22} + c_{33} + 2c_{23})^{-1}(c_{22}c_{33} - c_{23}^2) = & \\
 & = (c_{11} + c33 + 2c_{13})^{-1}(c_{11}c_{33} - c_{13}^2) = (c_{11} + c_{22} + 2c_{12})^{-1}(c_{11}c_{22} - c_{12}^2). &
\end{array}
$$

The case A is most simple but not useful for the purpose of approximating the slowness surface of one type of wave since the ellipsoidal symmetry appears as a result of crossing slowness surfaces of different type of waves. The solution for this case is given by (Ettrich et al., 2000).

We consider the case B. Here there are five independent parameters c_{11}, c_{22}, c_{33}, c_{44} and c_{12}. These parameters must be adjusted to best approximate an anisotropic medium by an ellipsoidal one. Taking the physics of wave propagation into account, we minimize the average of the sum of the differences between the Christoffel matrices $\Lambda_{il} = \lambda_{ijkl}n_j n_k$ of the anisotropic medium and $\Lambda_{il}^{(0)}$ of the ellipsoidal medium. Here, n_i is the i-component of the unit vector that points into the direction of the wave front propagation. For details, see (Fedorov, 1968) and (Ettrich et al., 2000).

For the case B with (4) the Christoffel matrix $\Lambda_{il}^{(0)}$ simplifies to:

$$
\Lambda^{(0)} = \begin{pmatrix} c_{11}n_1^2 + c_{66}n_2^2 + c_{44}n_3^2 & (c_{12} + c_{66})n_1 n_2 & 0 \\ (c_{12} + c_{66})n_1 n_2 & c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 & 0 \\ 0 & 0 & c_{44}(n_1^2 + n_2^2) + c_{33}n_3^2 \end{pmatrix}
$$

and

$$
< I >= \left\langle \sum_{i,l=1}^{3} \left(\Lambda_{il} - \Lambda_{il}^{(0)}\right)^2 \right\rangle = \left\langle \mathrm{tr}\left(\Lambda_{il}^2\right) \right\rangle - 2\left\langle \mathrm{tr}\left(\Lambda_{il}\Lambda_{il}^{(0)}\right) \right\rangle + \left\langle \mathrm{tr}\left(\left(\Lambda_{il}^{(0)}\right)^2\right) \right\rangle
$$

has to be minimized with respect to c_{11}, c_{22}, c_{33}, c_{44} and c_{12}. $< A >$ denotes averaging a function A:

$$
< A >= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A(\mathbf{n}) \sin\theta d\theta\phi,
$$

where $\mathbf{n} = \mathbf{n}(\theta, \phi)$ is a wavefront normal. This averaging is used in order to remove the dependency of the function on the direction ((Fedorov, 1968)). Since parameter c_{66} is a complicated function of c_{11}, c_{22} and c_{12} it is more convenient to consider c_{66} as an independent parameter and to seek the minimum of

$$
\Phi =< I > +\gamma\left((c_{11} + c_{22} + 2c_{12})^{-1}(c_{11}c_{22} - c_{12}^2) - c_{66}\right),
$$

with respect to c_{11}, c_{22}, c_{33}, c_{44}, c_{12} and c_{66}. γ is Lagrange's factor.

As a result we get the following nonlinear system for four unknown parameters c_{11}, c_{22}, c_{12} and $\tilde{\gamma}$:

$$\begin{aligned}
c_{12} &= D_2 + \tilde{\gamma}\left[E_1 - A\left(\frac{\partial f}{\partial c_{11}} + \frac{\partial f}{\partial c_{22}}\right) - E_2\frac{\partial f}{\partial c_{12}}\right] \\
c_{22} &= D_1 - G + \tilde{\gamma}\left[A - B\frac{\partial f}{\partial c_{11}} - C\frac{\partial f}{\partial c_{22}} - A\frac{\partial f}{\partial c_{12}}\right] \\
c_{11} &= D_1 + G + \tilde{\gamma}\left[A - C\frac{\partial f}{\partial c_{11}} - B\frac{\partial f}{\partial c_{22}} - A\frac{\partial f}{\partial c_{12}}\right] \\
\tilde{\gamma} &= \frac{77}{10}(c_{11} + c_{22} + 2c_{12})^{-1}(c_{11}c_{22} - c_{12}^2) - \frac{17}{20}(c_{11} + c_{22}) - 2c_{12} - \frac{77}{10}d_3
\end{aligned} \tag{5}$$

where:

$$A = \frac{17}{296}; \qquad B = \frac{23}{3\cdot 296}; \qquad C = \frac{319}{3\cdot 296}; \qquad E_1 = \frac{57}{296}; \qquad E_2 = \frac{205}{296}$$

$$D_1 = \frac{57e_2 + 17e_3}{296}; \qquad D_2 = \frac{17e_2 + 205e_3}{296}; \qquad G = \frac{e_1}{2}.$$

The remaining of parameters c_{33}, c_{44} and c_{66} can be linearly expressed by c_{11}, c_{22}, c_{12} and $\tilde{\gamma}$:

$$\begin{aligned}
c_{33} &= d_1 + \frac{3}{73}(c_{11} + c_{22}) - \frac{2}{73}c_{12} - \frac{1}{73}\tilde{\gamma} \\
c_{44} &= d_2 - \frac{9}{73\cdot 2}(c_{11} + c_{22}) + \frac{3}{73}c_{12} + \frac{3}{73\cdot 2}\tilde{\gamma} \\
c_{66} &= d_3 - \frac{8}{73}(c_{11} + c_{22}) - \frac{19}{73}c_{12} - \frac{19}{2\cdot 73}\tilde{\gamma}.
\end{aligned} \tag{6}$$

In (5) and (6) we used the abbreviations:

$$d_1 = \frac{27a_3 - 4a_4 + a_5}{73}; \qquad d_2 = \tfrac{1}{73}\Big(-4a_3 + 6a_4 - \tfrac{3}{2}a_5\Big); \qquad d_3 = \frac{1}{73}\Big(a_3 - \frac{3}{2}a_4 + \frac{19}{2}a_5\Big);$$

$$e_1 = a_1 - a_2; \qquad e_2 = a_1 + a_2 - 2(d_2 + d_3); \qquad e_3 = a_6 - 2d_3;$$

$$\begin{aligned}
\tilde{\gamma} &= \frac{15}{2}\gamma; \\
a_1 &= \lambda_{1kk1} + 2\lambda_{1111}; \\
a_2 &= \lambda_{2kk2} + 2\lambda_{2222}; \\
a_3 &= \lambda_{3kk3} + 2\lambda_{3333}; \\
a_4 &= \lambda_{1kk1} + 2\lambda_{1331} + \lambda_{2kk2} + 2\lambda_{2332} + \lambda_{3kk3} + 2\lambda_{3113} + \lambda_{3kk3} + 2\lambda_{3223}; \\
a_5 &= \lambda_{1kk1} + 2\lambda_{1221} + \lambda_{2kk2} + 2\lambda_{2112} + 4\lambda_{1122} + 4\lambda_{1212}; \\
a_6 &= 4(\lambda_{1122} + \lambda_{1212}); \\
f &= (c_{11} + c_{22} + 2c_{12})^{-1}(c_{11}c_{22} - c_{12}^2) - c_{66};
\end{aligned}$$

and:

$$\frac{\partial f}{\partial c_{11}} = (c_{11}+c_{22}+2c_{12})^{-1}\Big[c_{22}-(c_{11}+c_{22}+2c_{12})^{-1}(c_{11}c_{22}-c_{12}^2)\Big];$$
$$\frac{\partial f}{\partial c_{22}} = (c_{11}+c_{22}+2c_{12})^{-1}\Big[c_{11}-(c_{11}+c_{22}+2c_{12})^{-1}(c_{11}c_{22}-c_{12}^2)\Big];$$
$$\frac{\partial f}{\partial c_{12}} = -2(c_{11}+c_{22}+2c_{12})^{-1}\Big[c_{12}+(c_{11}+c_{22}+2c_{12})^{-1}(c_{11}c_{22}-c_{12}^2)\Big];$$

To approximate an anisotropic medium by an elliptical one, the non-linear system (5) should be solved by a numerical method. When four parameters c_{11}, c_{22}, c_{12} and $\tilde{\gamma}$ are computed the remaining parameters c_{33}, c_{44} and c_{66} are determined by (6).

Solving the non-linear system inside each cell of the FD-grid is too expensive. The proposed way, therefore, is suitable for a piece-wise homogeneous anisotropic medium when it is possible to use the elliptical reference medium for each layer. The ellipsoidal medium can be used for calculation of traveltimes by an FD eikonal solver with embedded perturbation scheme.

CONCLUSION

The presented algorithm provides a method for the efficient computation of the first arrival traveltimes for 3-D elliptically anisotropic media. The maximum relative error does not exceed 0.4 %. The calculation technique for approximating an arbitrary anisotropic medium by an ellipsoidal medium was considered. As a result we get the nonlinear system for defining parameters of a reference medium. Future work must be devoted to solving this system for embedding a first order perturbation method to the FD scheme in order to consider an arbitrary anisotropic medium. This technique is suitable for piece-wise homogeneous anisotropic medium.

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