Wave Inversion Technology, Report No. 4, pages 249-259

First arrival traveltimes and amplitudes by FD solution of eikonal and transport equation

S. Buske[1]

keywords: *modeling, FD, first arrival, eikonal equation, transport equation*

ABSTRACT

This paper describes the development and the implementation of a combined first-arrival traveltime and amplitude computation procedure. The schemes described here are based on a finite difference solution of the eikonal equation and the transport equation. After formulation of both equations as hyperbolic conservation laws the eikonal equation is solved numerically by a third-order ENO-Godunov scheme for the traveltimes and the transport equation is solved by a first-order upwind scheme for the amplitudes. The schemes are implemented in cartesian coordinates for a plane wave and in polar coordinates for a point source, respectively. The resulting traveltimes are highly accurate and amplitudes are smooth even in the case of complex models, for instance the Marmousi model.

INTRODUCTION

First-arrival traveltimes are still a popular and widely used tool for prestack Kirchhoff migration and tomographic applications. A variety of methods have been developed to solve the eikonal equation in order to efficiently obtain these traveltimes. (Vidale, 1988, 1990) has been one of the first to approximate the eikonal equation by finite differences and to solve it along expanding rectangles around the source. This method is very efficient and can be coded easily but it may fail to compute the correct solution for certain models. (Podvin and Lecomte, 1991) used a similar FD approximation, which incorporates head waves and diffracted waves. Due to its specific implementation this method is unconditionally stable but less efficient than Vidale's scheme. Another way of obtaining first arrivals are the so called 'shortest path' methods which are mainly based on geometric considerations (Moser, 1991; Klimes and Kvasnicka, 1994). The most interesting alternative in terms of the paper presented here has been published by (van Trier and Symes, 1991) for 2-D and (Schneider, 1995) for 3-D models. In this case the eikonal equation is formulated as a hypberbolic conservation

[1]**email:** buske@geophysik.fu-berlin.de

law (HCL) and is solved numerically by the corresponding schemes. One of the most recent papers which uses the HCL-approach has been published by (Sethian and Popovici, 1999). They treat the eikonal equation as a generalized level-set equation and solve it numerically by a fast marching procedure. At present this variant seems to be the most efficient way of obtaining first-arrival traveltimes in 3-D. A good overview and a comparison between different eikonal solvers covering the period up to 1998 can be found in (Leidenfrost et al., 1999).

All the methods mentioned above are more or less based on the numerical solution of the eikonal equation. The relation of the resulting first-arrival traveltimes with the respective amplitudes is non-trivial and has been subject of only a few investigations. (Vidale and Houston, 1990) tried to estimate geometrical spreading factors with the help of additional traveltime computations for sources surrounding the actual source location. The resulting errors are large and could only be reduced somewhat by an improved scheme published later (Pusey and Vidale, 1991). The first serious attempt was made by (El-Mageed, 1996) who followed the HCL-approach and solved the transport equation for the amplitudes in addition to the eikonal equation for the traveltimes. For the numerical calculation she used second order schemes for both equations, however, the resulting amplitudes were of zeroth order because they depend on the second traveltime derivatives. The method presented here is mainly based on the work of El-Mageed and extends as well as improves it by using alternative schemes resulting in highly accurate traveltimes and reliable amplitudes.

BASICS

In this chapter the eikonal and transport equation are introduced by starting from the equation of motion and considering acoustic waves propagating in a 2-D inhomogeneous isotropic medium. Then a high frequency approximation yields the eikonal equation

$$\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2 = s^2 \qquad s = \frac{1}{\alpha} \tag{1}$$

and the transport equation

$$\frac{\partial B}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial B}{\partial z}\frac{\partial T}{\partial z} + \frac{1}{2}\left[\frac{\partial \ln \frac{1}{\rho}}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial \ln \frac{1}{\rho}}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right] = 0. \tag{2}$$

x and z are the independent spatial variables, ρ is the density and α is the compression wave velocity. The inverse velocity is called slowness s. T stands for the traveltime and B stands for the logarithmic amplitude A

$$B = \ln A. \tag{3}$$

For a given density and velocity function the general procedure would be to first solve the eikonal equation for the traveltimes T and then use these traveltimes and solve the transport equation for the logarithmic amplitude B (or the amplitude A).

METHOD

In this chapter the procedure used to solve the eikonal and the transport equation is described. This part of the paper can only be regarded as a summary, for an introduction into hyperbolic conservation laws see (LeVeque, 1992) and for a detailed description of the method used here see (Buske, 2000).

Eikonal equation

The first step is to introduce as a new variable u the horizontal slowness and to rewrite the eikonal equation

$$\frac{\partial u}{\partial z} + \frac{\partial H(u)}{\partial x} = 0 \qquad u \equiv \frac{\partial T}{\partial x} \tag{4}$$

This equation has the form of a hyperbolic conservation law with the so called flux function H

$$H(u) = -\sqrt{s^2 - u^2} \quad . \tag{5}$$

The second step is to solve this equation for the traveltime function T. We assume a wave starting at $z = 0$ and propagating in positive z-direction. In this case for the numerical solution it is more convenient to write the eikonal equation in the form

$$\frac{\partial T}{\partial z} = -H(u) \tag{6}$$

with the traveltime given at $z = 0$

$$T(x, z = 0) = T_0(x)\,. \tag{7}$$

$T_0(x)$ describes the shape of the wave front at $z = 0$, a plane wave would be simply $T_0(x) = 0$. Introducing the finite difference notation for a cartesian grid with indices ix and iz and the constant grid intervals $\triangle x$ and $\triangle z$

$$x = (ix - 1)\,\triangle x \qquad ix = 1, ..., nx \qquad z = (iz - 1)\,\triangle z \qquad iz = 1, ..., nz \tag{8}$$

and replacing the derivative of T with respect to z by a simple first-order forward difference operator yields

$$T_{ix,iz+1} = T_{ix,iz} - \triangle z H(u)\,. \tag{9}$$

This form allows in principle the successive computation of traveltimes from level (iz) to $(iz + 1)$ for all grid points in the model. As pseudo-code this would look as follows:

1. initialise plane wave
   do $ix = 1, nx$
   $\quad T_{ix,1} = 0$
   end do

2. extrapolate forward
   do $iz = 1, nz - 1$
   $\quad$ do $ix = 1, nx$

   - compute u (derivative of T with respect to x) from known traveltimes at level iz by (for instance) a central difference operator
   - compute flux function
     $H(u) = -\sqrt{s^2_{ix,iz} - u^2}$
   - compute new traveltime at level $iz + 1$
     $T_{ix,iz+1} = T_{ix,iz} - \triangle z H(u)$

   $\quad$ end do
   end do

Unfortunately, this simple approach fails. For instance kinks in the wavefront will introduce unstable oscillations if a central difference operator is used to compute the value of u. This problem itself is well known in the literature on the numerical solution of hyperbolic conservation laws. Fortunately, a variety of numerical methods have been developed which avoid this problem. The main idea is to use one-sided difference operators for the computation of u and then to decide which is the correct one in terms of the propagation direction of the wave. For a detailed discussion of this subject see (Buske, 2000). Here, only result of this approach is presented.

First, compute left-sided and right-sided differences of third order ($u_l = D_x^{-,3}$ and $u_r = D_x^{+,3}$) with the ENO-scheme (Osher and Shu, 1991). Then, use the Godunov method to decide which of the two differences gives the correct solution and compute the flux function for the selected value of u

$$H_{GOD}(u_l, u_r) = \left\{ \begin{array}{lll} \min_{u\in[u_l,u_r]} H(u) & \text{for} & u_l \le u_r \\ \max_{u\in[u_r,u_l]} H(u) & \text{for} & u_l > u_r \end{array} \right\} . \tag{10}$$

The Godunov scheme is simply a selection criterion which chooses the left-sided difference if the wave propagates from the left to the right and vice versa (for more complex cases see (Buske, 2000)). From the numerical point of view the following is a slightly more convenient form:

$$H_{GOD}(u_l, u_r) = H(\text{maxmod}(\max(u_l, 0), \min(u_r, 0))), \tag{11}$$

$$\text{maxmod}(a, b) = \left\{ \begin{array}{lll} a & \text{for} & |a| > |b| \\ b & \text{else} & \end{array} \right\} . \tag{12}$$

Finally, replace the simple first-order forward difference for the z derivative of T by the Runge-Kutta-type formulas of Heun (Bronstein and Semendjajew, 1987) in order to build a consistent third-order scheme in x-direction as well as in z-direction

$$k_1 \equiv -\triangle z H_{GOD}\left(D_x^{-,3} T_{ix,iz}, D_x^{+,3} T_{ix,iz}\right) \quad \text{with} \quad s^2(x, z) \tag{13}$$

$$k_2 \equiv -\triangle z H_{GOD}\left(D_x^{-,3}(T_{ix,iz} + k_1), D_x^{+,3}(T_{ix,iz} + k_1)\right) \quad \text{with} \quad s^2(x, z+\triangle z) \tag{14}$$

$$k_3 \equiv -\triangle z H_{GOD}\left(D_x^{-,3}(T_{ix,iz} + \frac{1}{4}(k_1 + k_2)), D_x^{+,3}(T_{ix,iz} + \frac{1}{4}(k_1 + k_2))\right) \quad \text{with} \quad s^2(x, z + \frac{\triangle z}{2}) \tag{15}$$

$$T_{ix,iz+1} = T_{ix,iz} + \frac{1}{6}k_1 + \frac{1}{6}k_2 + \frac{2}{3}k_3 \, . \tag{16}$$

Note that in principle no changes have to be made to the pseudo-code described above.

The procedure described here is an explicit finite difference scheme and for that reason a stability condition applies, which in this case connects the step-size $\triangle z$ with the grid interval $\triangle x$ and the derivatives of the traveltime function

$$\triangle z < \left| \triangle x \frac{\frac{\partial T}{\partial z}}{\frac{\partial T}{\partial x}} \right| \quad . \tag{17}$$

Transport equation

For the transport equation (right side abbreviated by variable a)

$$\frac{\partial B}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial B}{\partial z}\frac{\partial T}{\partial z} = -\frac{1}{2}\left[\frac{\partial \ln \frac{1}{\rho}}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial \ln \frac{1}{\rho}}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right] \equiv -a \tag{18}$$

things are similar. Now, the new variable u refers to the derivative of the logarithmic amplitude B with respect to x and the transport equation in the form of a hypberbolic conservation law reads

$$\frac{\partial u}{\partial z} + \frac{\partial G(u)}{\partial x} = 0 \qquad u \equiv \frac{\partial B}{\partial x} \tag{19}$$

with the flux function

$$G(u) = \left(\frac{\partial T}{\partial z}\right)^{-1}\left(a + \frac{\partial T}{\partial x}u\right). \tag{20}$$

The corresponding equations which allow the successive numerical computation on the grid are

$$\frac{\partial B}{\partial z} = -G(u) \tag{21}$$

with for instance a constant amplitude at $z = 0$

$$B(x, z = 0) = 0\,. \tag{22}$$

In the case of the transport equation a first-order scheme for the forward extrapolation as well as the computation of the derivatives of B with respect to x is sufficient and reads

$$B_{ix,iz+1} = B_{ix,iz} - \triangle z G_{UPW}\left(u_l, u_r\right)\ , \tag{23}$$

where the flux function is computed using a simple upwind scheme

$$G_{UPW}(u_l, u_r) = \left(\frac{\partial T}{\partial z}\right)^{-1} \left(a + \max\left(\frac{\partial T}{\partial x}, 0\right) u_l + \min\left(\frac{\partial T}{\partial x}, 0\right) u_r\right) . \qquad (24)$$

This upwind scheme behaves in the same way as the Godunov scheme such that it chooses the correct difference operator depending on the propagation direction of the wave. The procedure itself is equivalent to the computation of the traveltimes and can actually be combined; for details see (Buske, 2000), too.

Remarks

Here, the method has been formulated for a cartesian grid and a plane wave starting at $z = 0$ and propagating in positive z-direction. For a point source the formulation and implementation is straightforward and can be found in (Buske, 2000). Polar coordinates centered at the source are used and the cartesian coordinates x and z are replaced by the angle φ and the radius r. The computation proceeds on circles around the source location in positive r-direction. If necessary a final interpolation to a cartesian grid is performed in order to use the traveltimes and amplitudes for instance in Kirchhoff prestack migration.

APPLICATION TO MARMOUSI MODEL

The method has been tested on various analytic models and has proven to perform very efficiently and to yield highly accurate traveltimes and reliable amplitudes. Here, the application to a slightly smoothed version of the well known and widely used Marmousi model (Versteeg and Grau, 1991) is presented.

Computations were performed for a point source at ($x = 6$ km, $z = 0$ km) and the parameters for the polar discretization were $\triangle r = 5$m and $\triangle\varphi = \pi/200$. Figure 1 shows the results as grey-scaled amplitudes and traveltime isochrons. As expected amplitudes are high near the source and decrease with increasing distance from the source according to geometrical spreading. In regions where the wave focuses and the wave front curvature is high amplitudes increase as well. Figure 2 again shows the traveltime isochrons but now overlayed by those computed with a wave front construction code (Buske and Kästner, 1999). The agreement is very good even at large distances from the source and in regions where wavefront kinks exist. Finally, Figure 3 shows the comparison of the amplitudes with geometrical spreading factors computed by an alternative method which is based on the ray propagator (Gajewski, 1998; Kästner and Buske, 1999). The general amplitude distribution is similar to the FD solution of the transport equation. Nevertheless, the result of the ray-propagator-method is noisier and is superimposed by distortions at isolated gridpoints.

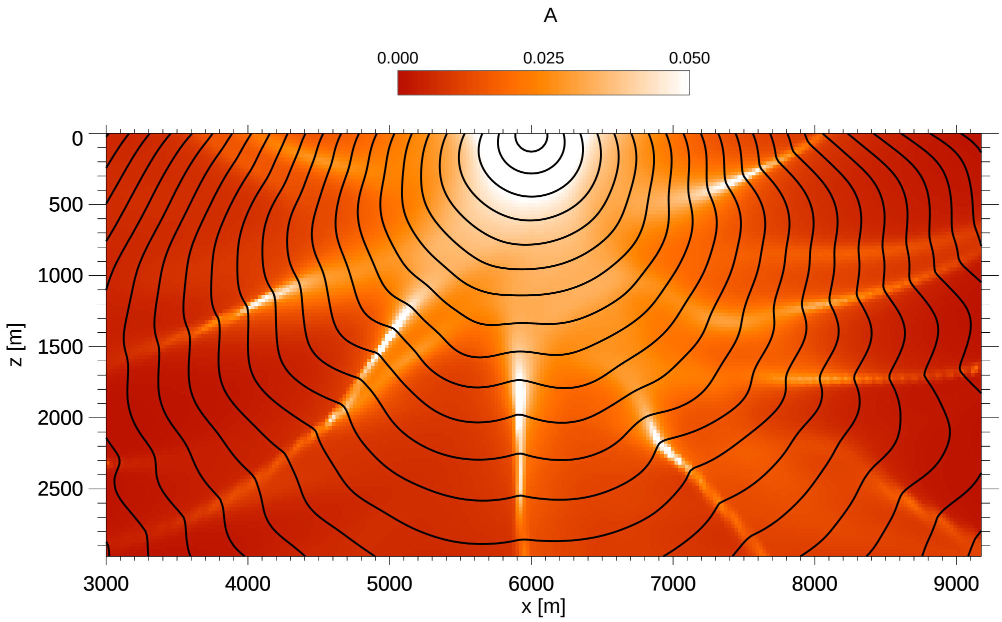

Figure 1: Traveltimes and amplitudes for the Marmousi model computed with the FD solution of the eikonal and the transport equation. The distance between isochrons is 0.075 seconds.

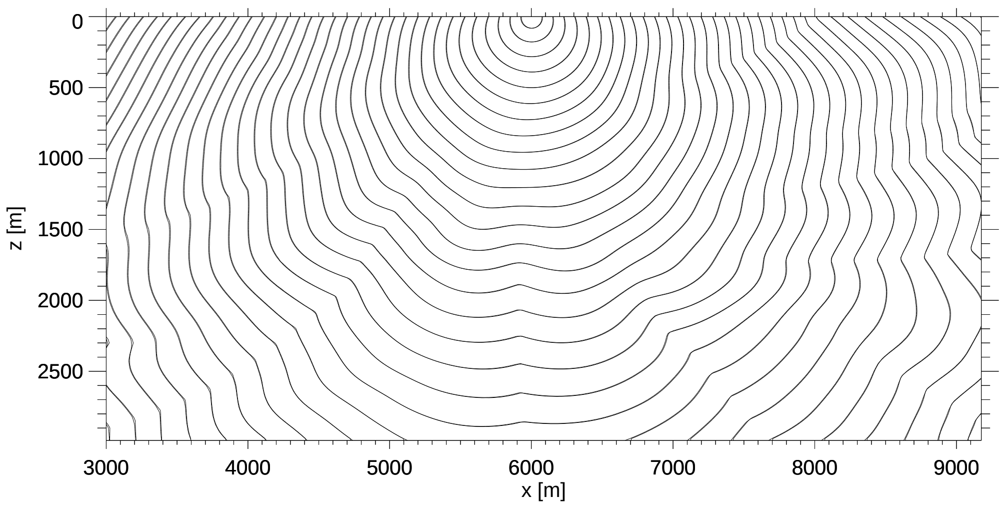

Figure 2: Traveltimes for the Marmousi model computed with the finite difference solution of the eikonal equation and a wave front construction program. The distance between isochrons is 0.05 seconds.

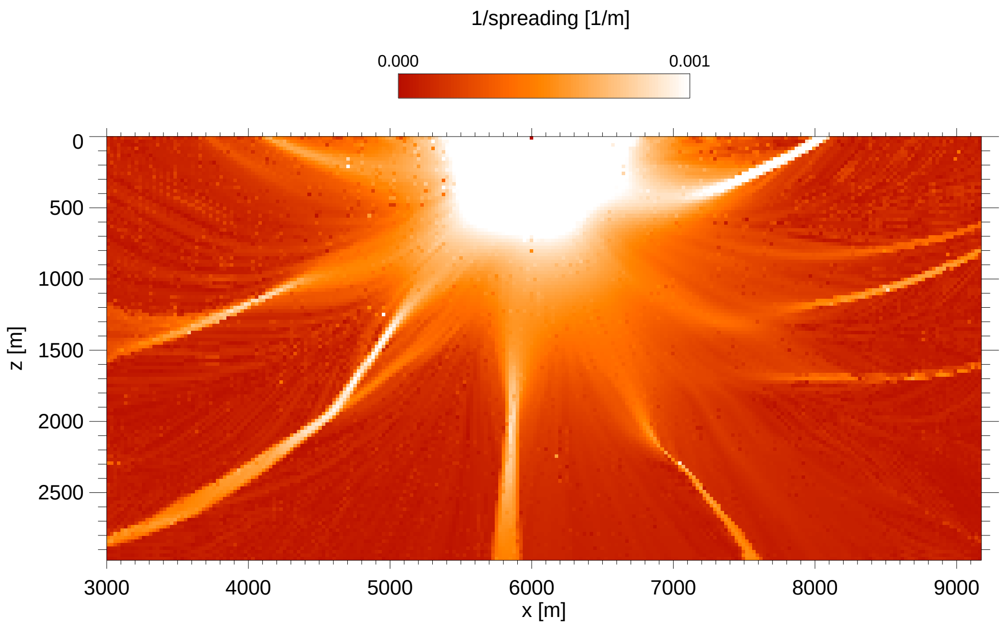

Figure 3: Inverse spreading values for the Marmousi model computed with the ray-propagator-method.

CONCLUSION

A combined first-arrival traveltime and amplitude computation procedure has been presented. The method is based on the formulation of the equations as hyperbolic conservation laws and the numerical solution with finite differences. The eikonal equation is solved by a third-order ENO-Godunov scheme for the traveltimes and the transport equation is solved by a first-order upwind scheme for the amplitudes. The schemes are implemented in cartesian coordinates for a plane wave and in polar coordinates for a point source, respectively. The application to the Marmousi model shows that even in the case of such a complex model the method proposed here yields highly accurate traveltimes and reliable and smooth amplitudes. The formulation in 3-D is straightforward.

REFERENCES

Bronstein, I., and Semendjajew, K., 1987, Taschenbuch der Mathematik: Verlag Harri Deutsch.

Buske, S., and Kästner, U., 1999, A practical approach to the computation of multivalued geometrical spreading factors: 61st EAGE meeting, Helsinki, Expanded Abstracts.

Buske, S., 2000, Finite difference solution of the raytheoretical transport equation: Ph.D. thesis, University of Frankfurt.

El-Mageed, M., 1996, 3D first arrival traveltimes and amplitudes via eikonal and transport finite difference solvers: Ph.D. thesis, Rice University, Department of Computational and Applied Mathematics, Houston, Texas, USA.

Gajewski, D., 1998, Determining the ray propagator from traveltimes: 68th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts.

Kästner, U., and Buske, S., 1999, Computing geometrical spreading from traveltimes in 3-D: 61st EAGE meeting, Helsinki, Expanded Abstracts.

Klimes, L., and Kvasnicka, M., 1994, 3-D network ray tracing: Geophys. J. Int., **116**, 726–738.

Leidenfrost, A., Ettrich, N., Gajewski, D., and Kosloff, D., 1999, Comparison of six different methods for calculating traveltimes: Geophysical Prospecting, **47**, 269–297.

LeVeque, R., 1992, Numerical methods for conservation laws: Birkhäuser.

Moser, T., 1991, Shortest path calculation of seismic rays: Geophysics, **56**, 59–67.

Osher, S., and Shu, C.-W., 1991, High-order essentially non-oscillatory schemes for Hamilton-Jacobi equations: SIAM J. Numer. Anal., **28**, 907–922.

Podvin, P., and Lecomte, I., 1991, Finite difference computation of traveltimes in very contrasted velocity models: a massively parallel approach and its associated tools: Geophys. J. Int., **105**, 271–284.

Pusey, L., and Vidale, J., 1991, Accurate finite-difference calculation of WKBJ traveltimes and amplitudes: 56th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1513–1516.

Schneider, W., 1995, Robust and efficient upwind finite-difference traveltime calculations in three dimensions: Geophysics, **60**, 1108–1117.

Sethian, J., and Popovici, M., 1999, 3-D traveltime computation using the fast marching method: Geophysics, **64**, 516–523.

van Trier, J., and Symes, W., 1991, Upwind finite-difference calculation of traveltimes: Geophysics, **56**, 812–821.

Versteeg, J., and Grau, G., 1991, Practical aspects of seismic data inversion, the Marmousi experience: EAEG workshop on practical aspects of seismic data inversion, Proceedings.

Vidale, J., and Houston, H., 1990, Rapid calculation of seismic amplitudes: Geophysics, **55**, 1504–1507.

Vidale, J., 1988, Finite-difference calculation of travel times: Bull. seism. Soc. Am., **78**, 2062–2076.

Vidale, J., 1990, Finite-difference calculation of traveltimes in three dimensions: Geophysics, **55**, 521–526.