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An asymptotic inverse to the Kirchhoff-Helmholtz integral

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ABSTRACT

The Kirchhoff-Helmholtz integral is a widely used tool for modeling the reflected response from an acoustic or elastic interface due to a given incident field. This integral is based on the idea that the reflection wavefield at the receiver is a superposition of the wave fields produced by point diffractors distributed at the reflecting interface. These diffractors are the so called Huygens secondary sources that are excited by the incident field. For a dense distribution of source-receiver pairs on a fixed measurement surface, the Kirchhoff-Helmholtz modeling integral provides, for a given interface, the corresponding reflection response as an amplitude distribution along a traveltime surface. The proposed asymptotic inverse Kirchhoff-Helmholtz integral consists of an integral along this reflection traveltime surface. For a point on the reflector, it sums the reflected-wave contributions attached to the respective reflection-traveltime surface associated with the related source-receiver pair. The new inverse integral reconstructs the Huygens sources along the reflector, thus providing their positions and amplitudes. In this way, a kinematic (positioning) and dynamic (amplitude) inversion becomes possible by means of an integral operation, which is most naturally related to its counterpart Kirchhoff-Helmholtz integral.

INTRODUCTION

The wavefield originating from a point source and primarily reflected from a smooth reflector overlain by a smooth inhomogeneous acoustic medium can be described by the Kirchhoff integral in the so-called single-scattering, high-frequency approximation (see, e.g.Frazer and Sen (1985)). The resulting Kirchhoff-Helmholtz integral describes then the reflected elementary waves as a superposition of Huygens secondary point sources distributed along the reflector.

The Kirchhoff-Helmholtz integral is largely used to accurately model primary re-

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flections in smoth layered models bounded by smooth interfaces (reflectors). A natural question that arises is whether a transformation exists that performs the opposite task of the Kirchhoff-Helmholtz integral. In other words, this inverse would have to *kinematically and dynamically reconstruct* the reflector. This would have to involve a weighted superposition of the observed elementary wave along the reflection traveltime surface of the searched-for reflector. To kinematically and dynamically reconstruct the reflector means to asymptotically recover the reflector location together with the plane-wave reflection coefficient in each point of the reflector. In the seismic literature, this is commonly called the *true amplitude* at all reflector points.

The depth migration method traditionally accepted as an inverse to the Kirchhoff-Helmholtz integral is Kirchhoff depth migration (Schneider, 1978). This migration is realized upon summing up contributions of the reflection data along auxiliary diffraction surfaces constructed on an a priori given reference model.

We see that the Kirchhoff-Helmholtz integral, a summation operator along a given reflector, lacks a structurely similar (asymptotic) inverse operation. This should have the form of a summation operation along the reflection traveltime corresponding the reflector, assuming, of course, the same configuration of source-receiver pairs. This is being set up in this paper by exploring the dual properties between the given reflector and its corresponding traveltime surface.

THE INVERSE KIRCHHOFF-HELMHOLTZ INTEGRAL

To set up an analogous integral to the forward Kirchhoff-Helmholtz integral (Frazer and Sen, 1985; Tygel et al., 1994) that achieves its inverse task, namely to reconstruct the singular function of the reflector Σ from its image at Γ, our strategy will be to substitute in the Kirchhff-Helmholtz (KH) integral all points and surfaces by their respective duals (Tygel et al., 1999). This is geometrically described with the help of Figure 1.

The new KH inverse will consist of an integration along the reflection-traveltime surface Γ, as opposed to its KH forward counterpart that involves an integration along the reflector Σ. In analogy with the preceding construction, we consider the output of the integration at a certain, fixed coordinate $\boldsymbol{x} = \vec{x}_R$, which determines a point $M_R = M_\Sigma(\vec{x}_R)$ on Σ. By means of a reflection ray, it also determines a dual point $N_R = N_\Sigma(\vec{x}_R) = N_\Gamma(\vec{\xi}_R)$ on Γ. Here, the value of the parameter $\vec{\xi}_R$ is given by $\vec{\xi}_R = \vec{\xi}_S(\vec{x}_R)$. The isochron of N_R will be tangent to Σ at $M_R = M_\Gamma(\vec{\xi}_R)$.

For each point N_Γ on Γ, the new integrand should contribute to the output result $I(\vec{x}_R, z)$ at a single point M_I, namely the intersection between the isochron of N_Γ, $z = \mathcal{Z}(\boldsymbol{x}, N_\Gamma)$, and the vertical line at $\vec{x}_R$. In symbols, $M_I = (\vec{x}_R, \mathcal{Z}(\vec{x}_R, N_\Gamma))$.

The point M_I will fall on Σ, i.e., it will coincide with M_R, when N_Γ coincides with

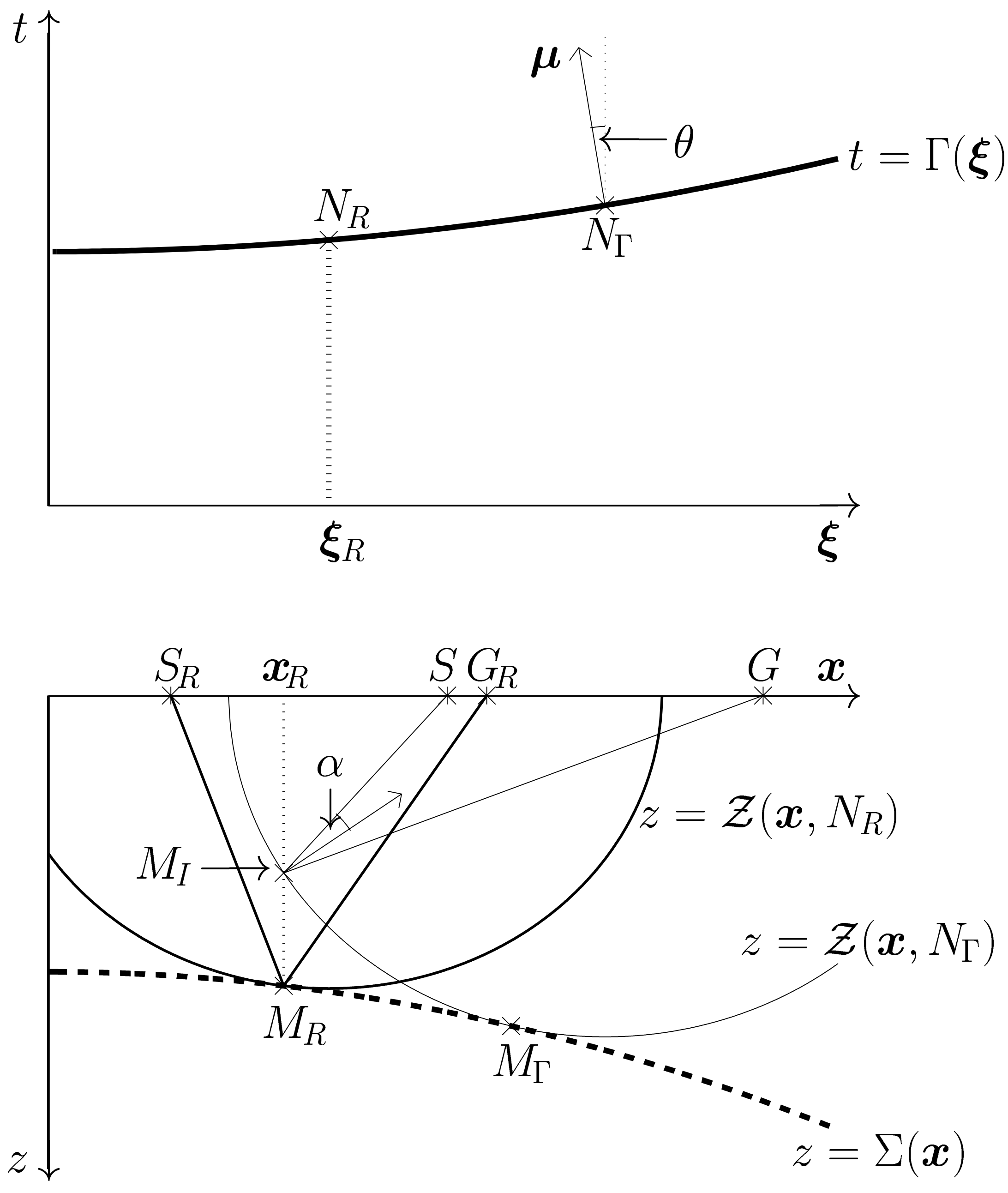


Figure 1: The inverse Kirchhoff-Helmholtz integral understood geometrically. For each point N_Γ on Γ, the integration contributes to the reflector depth image computed for $\vec{x}_R$ at the corresponding point $M_I = (\vec{x}_R, \mathcal{Z}(\vec{x}_R, N_\Gamma))$. For details see text.

N_R, the dual point of M_R. At M_R, the isochron $z = \mathcal{Z}(\boldsymbol{x}, N_R)$ is tangent to Σ. Due to our assumptions of a smooth reflector and uniqueness of dual points, we have again the situation of an isolated stationary point at M_R, which means that the main contribution of the new integral will be observed at M_R. As before, the reflector Σ (dashed line in Figure 1) is a priori unknown. It will become known only by a repeated execution of the above process for all $\boldsymbol{x}$ in E. In this way, we have geometrically constructed a kinematic transformation of the reflection-traveltime function Γ into the reflector Σ.

In the same way as the amplitude of the function to be integrated in the forward Kirchhoff-Helmholtz integral is that of the singular function $I_\Sigma(\boldsymbol{x}, z)$ of the reflector, multiplied with an appropriate weight function, it is most natural to set the amplitude of the integrand in the inverse integral to be that of the singular function $K_\Gamma(\vec{\xi}, t)$ of the reflection-traveltime surface Γ, multiplied with a corresponding, yet unspecified weight function. Analogously to the weight function $\mathcal{W}_K(\vec{\xi}, M_\Sigma)$ in the forward KH integral, the new weight function, $\mathcal{W}_I(\boldsymbol{x}, N_\Gamma)$, will be included into the inverse integral in order to assure that also this inverse transformation can be performed in a dynamically correct way, i.e., to correctly reconstruct the varying reflection coefficient along the reflector Σ. Similarly to the forward KH integral, the correct pulse shape will be ensured by the use of a derivative of the involved δ-pulse in the direction normal to Γ.

Translating the above arguments into mathematical terms and in full correspondence to the forward KH integral, we can now set up the proposed inverse KH integral as

$$I(\boldsymbol{x}, z) = -\frac{1}{4\pi} \int d\Gamma \; \mathcal{W}_I(\boldsymbol{x}, N_\Gamma) \; \mathcal{A}(N_\Gamma) \; \partial_\mu \delta(z - \mathcal{Z}(\boldsymbol{x}, N_\Gamma)) \; . \tag{1}$$

In this formula, ∂_μ denotes, correspondingly to the normal derivative ∂_η in the forward KH integral, the partial derivative in the direction of the normal $\vec{\mu}$ to the traveltime surface Γ at the generic point $N_\Gamma = (\vec{\xi}, \Gamma(\vec{\xi}))$ that describes the integral. The point N_Γ, by means of the configuration parameter $\vec{\xi}$, automatically determines the source-receiver pair (S, G), where $S = S(\vec{\xi})$ and $G = G(\vec{\xi})$. As previously, let us denote by M_R the (unique) specular reflection point on the reflector pertaining to the source-receiver pair (S, G) defined by $\vec{\xi}$. From a stationary-phase analysis, it follows that the weight function can be selected as

$$\mathcal{W}_I(\boldsymbol{x}, N_\Gamma) = \frac{h_B(\vec{\xi}, M_I) \; v^3(M_I) \; \cos^2\theta(M_I)}{\cos^2\alpha(M_I)} \mathcal{L}_S(M_I) \mathcal{L}_G(M_I) \; . \tag{2}$$

Here, M_I is the point $(\boldsymbol{x}, z = \mathcal{Z}(\boldsymbol{x}, N_\Gamma))$, where the isochron $z = \mathcal{Z}(\boldsymbol{x}, N_\Gamma)$ of N_Γ cuts the vertical line at M_R (see Figure 1). Also, $v(M_I)$ is the medium velocity at M_I and $\mathcal{L}_S(M_I)$ and $\mathcal{L}_G(M_I)$ are the point-source geometrical-spreading factors along the ray segments SM_I and M_IG, respectively. Moreover, $\theta(M_I)$ represents the angle the normal to Γ at N_Γ makes with the vertical t-axis), and $\alpha(M_I)$ denotes the incidence angle that the incoming ray SM_I makes with the isochron normal at M_I (see Figure 1). Finally, $h_B(\vec{\xi}, M_I)$ is the modulus of the Beylkin determinant (Beylkin, 1985; Bleistein, 1987),

$$h_B(\vec{\xi}, M_I) = \begin{bmatrix} \boldsymbol{\nabla}\mathcal{T}(\vec{\xi}, M_I) \\ \partial_{\xi_1}\boldsymbol{\nabla}\mathcal{T}(\vec{\xi}, M_I) \\ \partial_{\xi_2}\boldsymbol{\nabla}\mathcal{T}(\vec{\xi}, M_I) \end{bmatrix} , \tag{3}$$

where $\boldsymbol{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_z)$ is the spatial gradient operator.

As shown by Tygel et al. (1999) the forward and inverse KH integrals can be asymptotically evaluated by the time-domain version of the standard stationary-phase method, as described, e.g., in Bleistein (1984). The KH forward integral can be approximated, for time values $t \approx \Gamma(\vec{\xi})$ as

$$K(\vec{\xi}, t) \approx K_\Gamma(\vec{\xi}, t) = \mathcal{A}(N_\Gamma)\ \delta(t - \Gamma(\vec{\xi})) , \tag{4}$$

namely, the true-amplitude singular function of the traveltime surface Γ.

Correspondingly, the new KH inverse integral 1 can be asymptotically approximated for spatial values $z \approx \Sigma(\boldsymbol{x})$ as

$$I(\boldsymbol{x}, z) \approx I_\Sigma(\boldsymbol{x}, z) = \mathcal{R}(M_\Sigma)\ \delta(z - \Sigma(\boldsymbol{x})) , \tag{5}$$

i.e., the true-amplitude singular function of the reflector Σ.

This means that integral 1 is the (asymptotic) inverse to the forward KH integral. In other words, the forward and inverse KH integrals form an asymptotic transform pair between the depth-domain true-amplitude singular function $I_\Sigma(\boldsymbol{x}, z)$ of the reflector Σ and its corresponding time-domain true-amplitude singular function $K_\Gamma(\vec{\xi}, t)$ of the traveltime surface Γ.

A SIMPLE NUMERICAL EXAMPLE

To verify the validity of the inverse Kirchhoff-Helmholtz integral 1, we have designed the following simple numerical experiment. A seismic common-offset experiment with a half-offset of h =500 m was simulated above the earth model depicted in Figure 2. It consists of two homogeneous acoustic layers with constant velocities of 4 km/s and 4.5 km/s, respectively, separated by a smooth interface in the form of a dome structure. The terminology “common-offset experiment” means that all source-receiver pairs involved are separated by the same fixed offset of $2h = 1000$ m. More specifically, the location of each source and each receiver can be expressed as $x_S = \xi - h$ and $x_G = \xi + h$, where parameter ξ is the midpoint coordinate, i.e., $\xi = (x_S + x_G)/2$.

Figure 3a shows the common-offset data as modeled by the forward KH integral. For an easier analysis, we have convolved the results with a Ricker wavelet of unit peak amplitude and a duration of 64 ms. Also indicated in Figure 3a is the reflection traveltime curve as the locus of the peak amplitudes of the time-symmetric Ricker

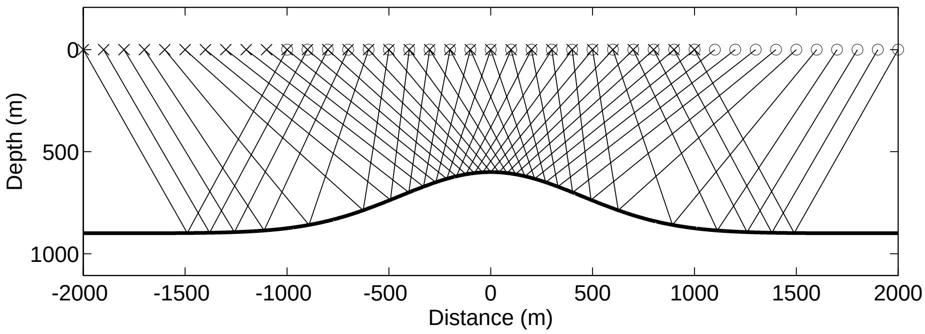


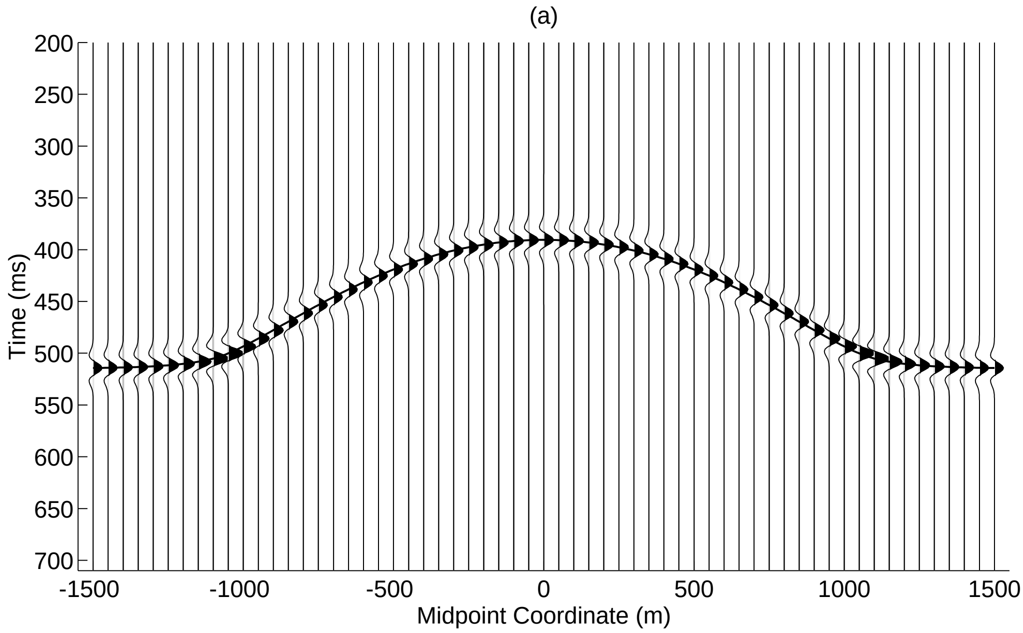
Figure 2: Model for the numerical example.

wavelet used in the numerical modeling. Along this identified traveltime curve, we now perform the new inverse Kirchhoff-Helmholtz integral 1. The result is shown in Figure 3b. The inversion results show that the obtained wavelets align perfectly along the reflector that is indicated by a continuous line. This confirms that integral 1 performs the inversion task in a kinematically correct way (i.e., it correctly positions the reflector in depth).

To check on the dynamics, we determine the obtained peak amplitudes and compare them to the correct reflection coefficient (see Figure 4a). We observe an almost perfect coincidence between the inverted amplitudes (green line) and the theoretical curve (black line) within the center region of the dome structure. On both flanks, a slightly larger error can be observed. This is due to the reduced illumination of this part of the reflector, caused by the employed measurement configuration. For comparison, also included are the amplitudes as recovered from conventional true-amplitude Kirchhoff migration (blue line). In Figure 4b, this observation is quantified by the relative error of the observed amplitudes. In the central region, the error of Kirchhoff inversion does not exceed half a percent. In the less well-illuminated parts of the reflector, we still have errors less than four percent, an also good amplitude recovery and still better than that of Kirchhoff migration. Note the smaller boundary zone of Kirchhoff inversion with respect to Kirchhoff migration. This indicates that Kirchhoff inversion will achieve a better horizontal resolution. These results (and other numerical experiments not shown in this paper) confirm our claim that integral 1 constitutes indeed an asymptotic inverse to the well-known forward Kirchhoff-Helmholtz integral.

CONCLUSIONS

We have presented an analogous (asymptotic) inverse to the well-known forward Kirchhoff-Helmholtz (KH) integral. Just as the forward KH integral can be conceived as a superposition of the elementary responses of all Huygens secondary sources along the



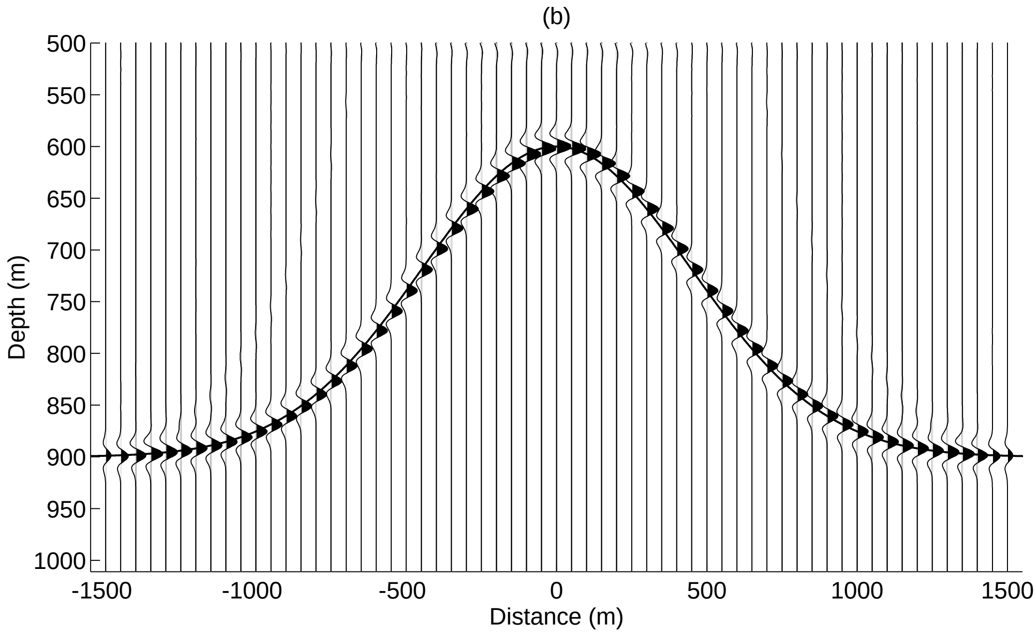


Figure 3: (a) Synthetic data as modeled by the forward Kirchhoff-Helmholtz integral. (b) Inverted data as obtained by the inverse Kirchhoff-Helmholtz integral.

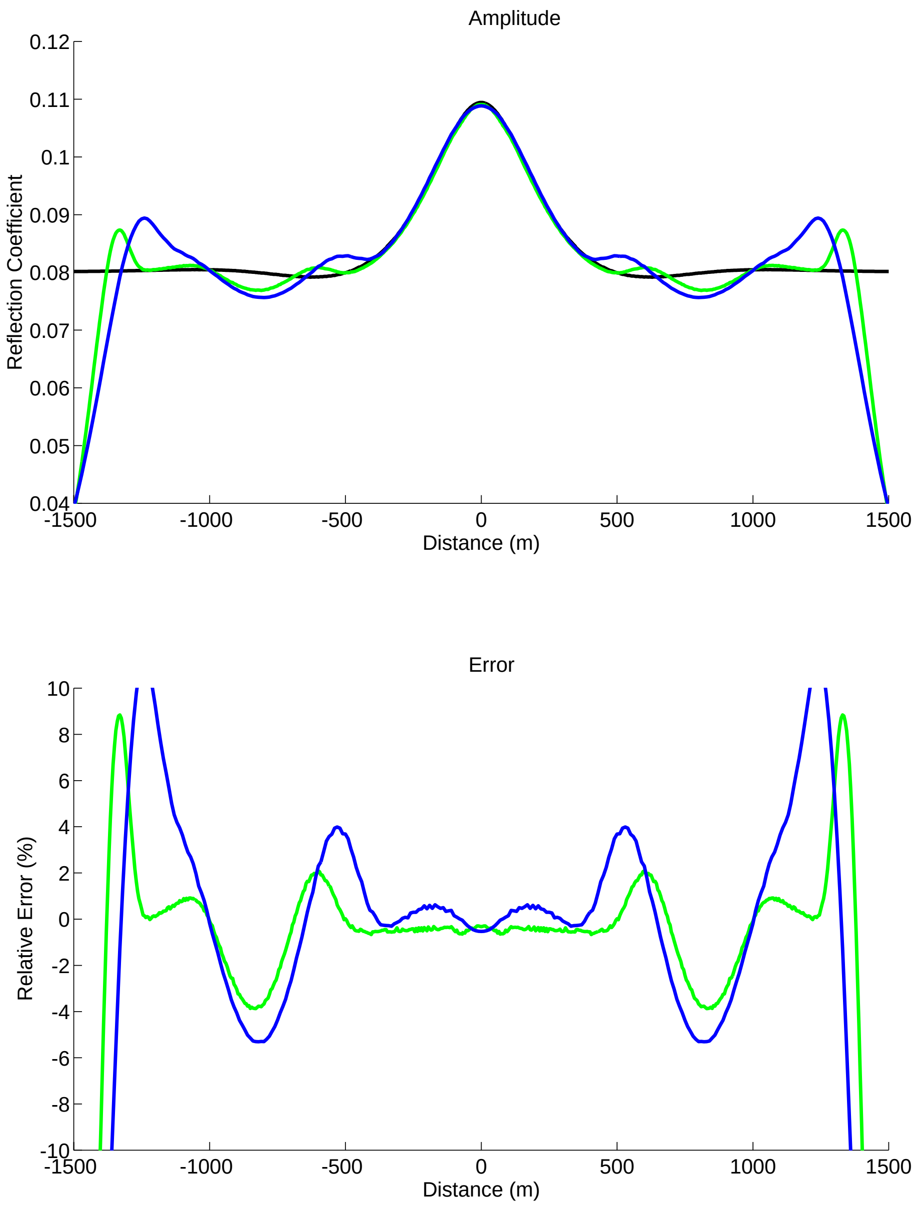


Figure 4: (a) Picked peak amplitude in the inverted section of Figure 3b (green line), as compared with the theoretical reflection coefficient (black line) and the corresponding amplitudes from standard true-amplitude Kirchhoff migration. (b) Relative error of the amplitudes of part (a).

reflector, we can conceive its inverse as a superposition of "elementary reflection images" along the reflection-traveltime surface. The elementary Huygens sources along the reflector are then recovered in position and strength.

For a given distribution of source-receiver pairs (the measurement configuration) that provides the "illumination" of the reflector, we have introduced the concepts of the true-amplitude singular functions of the reflector and its corresponding reflection-traveltime surface. These are nothing else than δ-functions localized on the two surfaces multiplied by specular reflection coefficients (at the reflector) and zero-order ray responses (at the reflection-traveltime surface). The new KH transform pair provides an (asymptotic) link between the true-amplitude singular functions of the reflector and its reflection-traveltime surface. The new inverse KH integral was constructed using the fundamental dual properties that relate primary-reflection surfaces of the time-domain data space and their corresponding depth-domain reflectors in model space. By a simple numerical example, we have confirmed that the new inverse Kirchhoff-Helmholtz integral indeed recovers the reflection coefficients along the reflecting interface as claimed theoretically.

The new inverse integral also fills a gap which originates from the observation that the conventional Kirchhoff migration integral (Schneider, 1978), well known in the seismic literature and frequently used to solve the inverse problem, is not an inverse but the adjoint operation to the forward KH integral (Tarantola, 1984). In fact, the structurely correct inverse to Kirchhoff migration has also been recently provided under the name of the Kirchhoff demigration integral (Hubral et al., 1996; Tygel et al., 1996).

The proposed inverse KH integral enables the design of a new seismic migration technique that would deserve the name Kirchhoff migration much more than what is up to now associated with this name. Note, however, that conventional Kirchhoff migration has done an excellent job in practice. Whether the new inverse Kirchhoff-Helmholtz integral can be employed with comparable success in practical seismic inverse problems remains a topic of future research.

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PUBLICATIONS

A paper containing these results has been submitted to Inverse Problems.