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Geometrical-spreading decomposition in anisotropic media

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ABSTRACT

The decomposition formula for the geometrical-spreading factor is generalized to anisotropic media. This involves not only the familiar ray-centered and local Cartesian coordinate system but also new coordinates oriented with respect to the group velocity. The general decomposition formula involves the two spreading factors of the two individual segments from the source to the reflector point and from there to the receiver, the Fresnel matrix, as well as the group velocity reflection angles and phase angles of the ray segments.

INTRODUCTION

In this paper, we derive the a geometrical spreading decomposition formula for anisotropic elastic media in terms of second-order mixed derivatives of the traveltime. This formula is crucial to the verification that the asymptotic evaluation of the Kirchhoff-Helmholtz integral provides the zero-order ray-theoretical response.

For didactical reasons, we divide the proof into two independent claims, the first being a general version of the decomposition formula for arbitrary local Cartesian coordinate systems having origins at the source and receiver locations, respectively. The second is the specification of the previous formula to ray-centered coordinates, thus producing the desired decomposition result for the relative geometrical spreading.

Referring to Figure 1, we consider a point source with global Cartesian coordinates $\boldsymbol{x}^s$ located on some arbitrary measurement surface, and a corresponding receiver at $\boldsymbol{x}^r$ on the same or another measurement surface. We also consider a fixed primary reflection ray that connects the source point $\boldsymbol{x}^s$ to the reflection point $\tilde{\boldsymbol{x}}$ on the reflector Σ and that reflects back to the receiver $\boldsymbol{x}^r$, in accordance to Snell's law.

We define a local 3-D Cartesian coordinate system $\hat{\boldsymbol{\sigma}}$ with its origin at $\tilde{\boldsymbol{x}}$. It is oriented such that its third axis, σ_3, is along the reflector's normal. The first two axes,

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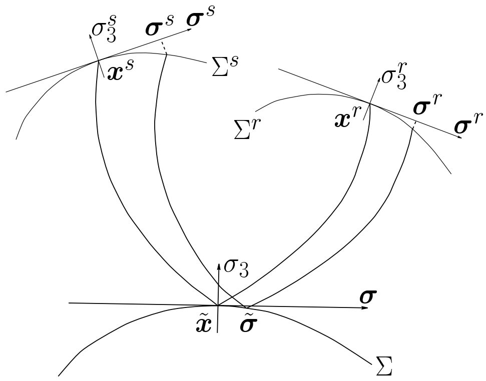

Figure 1: Global and local coordinates

σ_1, σ_2, then define a natural 2-D Cartesian coordinate system on the tangent plane to the reflector at $\tilde{\boldsymbol{x}}$. We distinguish between the 3-D vector $\hat{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2, \sigma_3)$ and the 2-D vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. We finally assume that points on the reflector in the vicinity of $\tilde{\boldsymbol{x}}$ are parameterized by their 2-D projection vectors $\boldsymbol{\sigma}$ on the tangent plane.

We introduce analogous 3-D Cartesian systems $\hat{\boldsymbol{\sigma}}^s$ and $\hat{\boldsymbol{\sigma}}^r$ with origins at $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$, respectively. For each of these, the third component points along the normal to each measurement surface at $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$, respectively. As above, the measurement surfaces are described locally as functions of the first two components, $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^r$, respectively.

We need to consider the traveltime from points on the source surface near $\boldsymbol{x}^s$ to points on the reflector near $\tilde{\boldsymbol{x}}$ to points on the receiver surface near $\boldsymbol{x}^r$. These traveltimes can be totally described as functions of the 2-D coordinate vectors, $\boldsymbol{\sigma}^s,\ \boldsymbol{\sigma},\ \boldsymbol{\sigma}^r$. That is,

$$T_D(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r, \boldsymbol{\sigma}) = T^s(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}) + T^r(\boldsymbol{\sigma}^r, \boldsymbol{\sigma}) \ . \tag{1}$$

This function is commonly referred to as the diffraction traveltime from the source to the reflector to the receiver. For each source-receiver pair, Fermat's principle tells us that the diffraction traveltime is stationary at the reflection point. That is,

$$\left.\frac{\partial T_D(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r, \boldsymbol{\sigma})}{\partial \sigma_i}\right|_{\boldsymbol{\sigma}=\tilde{\boldsymbol{\sigma}}} = 0 \qquad i, j = 1, 2 \ . \tag{2}$$

This determines the reflection point coordinates $\boldsymbol{\sigma}$ as a function of $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^r$. We call this function $\tilde{\boldsymbol{\sigma}}$: $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\sigma}^s, \sigma^r)$. In addition, this defines the reflection traveltime

$$T_R(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r) = T_D(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r, \tilde{\boldsymbol{\sigma}}(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r)) \tag{3}$$

of the ray from $\boldsymbol{\sigma}^s$ to $\tilde{\boldsymbol{\sigma}}$ to $\boldsymbol{\sigma}^r$. Due to the particular choice of the (3-D) coordinate systems $\hat{\boldsymbol{\sigma}}^s$, $\hat{\boldsymbol{\sigma}}^r$, and $\hat{\boldsymbol{\sigma}}$, the reflection traveltime along the ray from $\boldsymbol{x}^s$ to $\tilde{\boldsymbol{x}}$ to $\boldsymbol{x}^r$ is then given by

$$T_R(\boldsymbol{\sigma}^s = \mathbf{0}, \boldsymbol{\sigma}^r = \mathbf{0}) = T_D(\boldsymbol{\sigma}^s = \mathbf{0}, \boldsymbol{\sigma}^r = \mathbf{0}, \boldsymbol{\sigma} = \mathbf{0}) \; . \tag{4}$$

The mixed-derivatives of second-order of traveltimes play an important role in our analysis. More specifically, we are interested in matrices $\boldsymbol{B}$ related to these mixed derivatives by

$$B_{ij}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) = - \left. \frac{\partial^2 T_R(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r)}{\partial \sigma_i^s \partial \sigma_j^r} \right|_{\boldsymbol{\sigma}^s = \boldsymbol{\sigma}^r = \mathbf{0}} , \qquad i,j = 1,2 \; , \tag{5}$$

$$B_{ij}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) = - \left. \frac{\partial^2 T_R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^r)}{\partial \sigma_i \partial \sigma_j^r} \right|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}^r = \mathbf{0}} , \qquad i,j = 1,2 \; , \tag{6}$$

$$B_{ij}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = - \left. \frac{\partial^2 T_R(\boldsymbol{\sigma}^s, \boldsymbol{\sigma})}{\partial \sigma_i^s \partial \sigma_j} \right|_{\boldsymbol{\sigma}^s = \boldsymbol{\sigma} = \mathbf{0}} , \qquad i,j = 1,2 \; . \tag{7}$$

We are now ready to state the two independent claims that will provide the following decomposition formula for the geometrical spreading

$$|\det \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)|^{1/2} = \left| \det \boldsymbol{H} \; \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^r) \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \frac{\cos \chi^r \cos \chi^s}{\cos \alpha^r \cos \alpha^s} \right|^{1/2} . \tag{8}$$

Claim one

The first claim is that, for a reflection ray from a source at $\boldsymbol{x}^s$ to a receiver at $\boldsymbol{x}^r$, being reflected at $\tilde{\boldsymbol{x}}$, the matrix $\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s)$ can be decomposed into

$$\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{B}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \boldsymbol{H}(\tilde{\boldsymbol{x}}) \boldsymbol{B}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \; , \tag{9}$$

The matrix $\boldsymbol{H}(\tilde{\boldsymbol{x}})$ is defined in the present notation as

$$H_{ij}(\tilde{\boldsymbol{x}}) = \left. \frac{\partial^2 T_D(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r, \boldsymbol{\sigma})}{\partial \sigma_i \partial \sigma_j} \right|_{\boldsymbol{\sigma}^s = \boldsymbol{\sigma}^r = \boldsymbol{\sigma} = \mathbf{0}} , \qquad i,j = 1,2 \; . \tag{10}$$

The $\boldsymbol{B}$-matrix decomposition (9) has been demonstrated by Hubral et al. (1992b) based on paraxial-ray arguments. Here, we prove the same result independently using the simple rules of explicit differentiation. Although Hubral et al. were discussing isotropic media, both proofs are equally valid for anisotropic media.

Claim two

The second claim to be proven addresses the relationship between the 2×2 matrices $\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s)$ and $\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s)$. This has the form

$$\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{G}^{-1}(\boldsymbol{x}^r)\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s)\boldsymbol{G}^{-T}(\boldsymbol{x}^s) \ . \tag{11}$$

Here, $\boldsymbol{G}$ is the 2×2 transformation matrix from ray-centered to local Cartesian coordinates. In acoustic and isotropic media, Hubral et al. (1992a)) and Cervený (1995) have provided expressions for $\boldsymbol{G}$. Their results will now be extended to the anisotropic case.

Once the two claims (9) and (11) are proven, they will provide the decomposition formula for $\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s)$, the determinant of which is equation (8).

Proof of claim one

To prove the first claim, we start by using the chain rule to compute the second-order traveltime derivative required in equation (5). We have

$$\begin{aligned}
\frac{\partial^2 T_R(\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^r)}{\partial\sigma_i^s \partial\sigma_j^r} &= \frac{\partial}{\partial\sigma_i^s}\left(\left.\frac{\partial T_D}{\partial\sigma_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial T_D}{\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial\tilde{\sigma}_k}{\partial\sigma_j^r}\right) \\
&= \left.\frac{\partial^2 T_D}{\partial\sigma_i^s\partial\sigma_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} + \frac{\partial\tilde{\sigma}_l}{\partial\sigma_i^s}\left.\frac{\partial^2 T_D}{\partial\sigma_l\partial\sigma_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial^2 T_D}{\partial\sigma_i^s\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial\tilde{\sigma}_k}{\partial\sigma_j^r} \\
&\quad + \frac{\partial\tilde{\sigma}_l}{\partial\sigma_i^s}\left.\frac{\partial^2 T_D}{\partial\sigma_l\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial\tilde{\sigma}_k}{\partial\sigma_j^r} + \left.\frac{\partial T_D}{\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial^2\tilde{\sigma}_k}{\partial\sigma_i^s\partial\sigma_j^r}
\end{aligned} \tag{12}$$

The last term on the right-hand side vanishes because of equation (2). Also, the first term vanishes because neither T^s nor T^r in equation (1) depends on both $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^r$.

We now observe that

$$\frac{\partial^2 T_D}{\partial\sigma_i^s\partial\sigma_k} = \frac{\partial^2(T^s + T^r)}{\partial\sigma_i^s\partial\sigma_k} = \frac{\partial^2 T^s}{\partial\sigma_i^s\partial\sigma_k} = -B_{ik}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \ , \tag{13}$$

and

$$\frac{\partial^2 T_D}{\partial\sigma_l\partial\sigma_j^r} = \frac{\partial^2(T^s + T^r)}{\partial\sigma_l\partial\sigma_j^r} = \frac{\partial^2 T^r}{\partial\sigma_l\partial\sigma_j^r} = -B_{ik}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \ , \tag{14}$$

because $T^r = T^r(\boldsymbol{\sigma}^r, \sigma)$ does not depend on σ_i^s and $T^s = T^s(\boldsymbol{\sigma}^s, \sigma)$ does not depend on σ_j^r. Taking the above derivatives at $\boldsymbol{\sigma}^s = \boldsymbol{\sigma}^r = \boldsymbol{\sigma} = \boldsymbol{0}$, we can thus write equation (12) in matrix notation as

$$\boldsymbol{B}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)\boldsymbol{X}^r + (\boldsymbol{X}^s)^T\boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) - (\boldsymbol{X}^s)^T\boldsymbol{H}(\tilde{\boldsymbol{x}})\boldsymbol{X}^r \tag{15}$$

where the matrices $\boldsymbol{X}^s$ and $\boldsymbol{X}^r$ are defined by

$$X_{ij}^{s,r}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) = \left.\frac{\partial \tilde{\sigma}_i}{\partial \sigma_j^{s,r}}\right|_{\boldsymbol{\sigma}^{s,r}=\boldsymbol{\sigma}=\boldsymbol{0}} , \qquad i, j = 1, 2 . \tag{16}$$

Expressions for these matrices can be obtained by differentiating equation (2) with respect to σ_i^s and σ_i^r, respectively. This yields

$$\frac{\partial}{\partial \sigma_i^{s,r}} \left(\left.\frac{\partial T_D}{\partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \right) = \left.\frac{\partial^2 T_D}{\partial \sigma_i^{s,r} \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial^2 T_D}{\partial \sigma_l \partial \sigma_k}\right|_{\boldsymbol{\sigma}=\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_l}{\partial \sigma_i^{s,r}} = 0 , \tag{17}$$

which, evaluated at $\boldsymbol{\sigma}^s = \boldsymbol{\sigma}^r = \boldsymbol{\sigma} = \boldsymbol{0}$, translates in matrix notation to

$$\boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = (\boldsymbol{X}^s)^T \boldsymbol{H}(\tilde{\boldsymbol{x}}) \tag{18}$$

and

$$\boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) = \boldsymbol{H}(\tilde{\boldsymbol{x}}) \boldsymbol{X}^r . \tag{19}$$

Solving equations (18) and (19) for $\boldsymbol{X}^s$ and $\boldsymbol{X}^r$ and substituting the results in equation (15), we arrive at

$$\begin{aligned} \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) &= \boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \boldsymbol{H}^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \\ &\quad + \boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \boldsymbol{H}^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \\ &\quad - \boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \boldsymbol{H}^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{H}(\tilde{\boldsymbol{x}}) \boldsymbol{H}^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \\ &= \boldsymbol{B}^{-1}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \boldsymbol{H}^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) . \end{aligned} \tag{20}$$

Taking inverses of both sides of this equation reproduces the first claim (9).

Proof of claim two

To prove the second claim (11), we need to consider two additional auxiliary coordinate systems. These are the familiar ray-centered coordinates and a new set of coordinates that we will call the “group-velocity centered coordinates” (see Figure 2).

We denote the 3-D ray-centered coordinates by $\hat{\boldsymbol{q}}$, consistent with Cervený's (1995) notation. We recall that the vector $\boldsymbol{q}$ of the first two components of $\hat{\boldsymbol{q}}$ lies in the tangent plane to the wavefront, that is, the third axis, q_3, points along the slowness vector of the ray.

The group-velocity centered coordinate system, denoted by $\hat{\boldsymbol{g}}$, has the third axis, g_3, pointing along the group velocity, that is, in the direction of the ray.

As above, we define corresponding coordinate systems with superscripts s and r at the source and receiver positions $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$, respectively. Note that there are two ray-centered and two group-velocity centered coordinate systems at the stationary point,

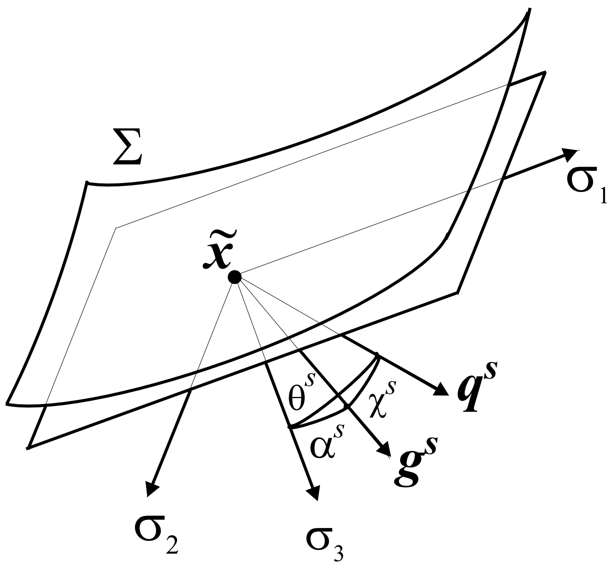

Figure 2: Local Cartesian, ray centered, and group-velocity centered coordinates

one of which pertains to the source ray and the other to the receiver ray, respectively. There is, however, only one local $\boldsymbol{\sigma}$-system at this point, because this system is not oriented with respect to any ray but to the reflecting interface.

Using these coordinates systems, we recall that (Cervený, 1995)

$$\boldsymbol{Q}_2^{-1}(\boldsymbol{x}^r,\boldsymbol{x}^s) = -\left(\frac{\partial^2 T_R}{\partial q_i^s \partial q_j^r}\right)\Bigg|_{\boldsymbol{q}=\mathbf{0}} , \qquad i,j=1,2 . \tag{21}$$

We will also need the corresponding traveltime mixed-derivative matrix in the group-velocity centered coordinates, viz.,

$$\boldsymbol{Y}^{-1}(\boldsymbol{x}^r,\boldsymbol{x}^s) = -\left(\frac{\partial^2 T_R}{\partial g_i^s \partial g_j^r}\right)\Bigg|_{\boldsymbol{q}=\mathbf{0}} , \qquad i,j=1,2 . \tag{22}$$

By the chain rule of partial derivatives, the above second derivative matrices are related by

$$\frac{\partial^2 T_R}{\partial q_i^s \partial q_j^r} = \frac{\partial g_k^s}{\partial q_i^s}\frac{\partial^2 T_R}{\partial g_k^s \partial g_l^r}\frac{\partial g_l^r}{\partial q_j^r} , \qquad i,j=1,2; \quad k,l=1,2,3 . \tag{23}$$

In the same way, the second-order derivatives defining the matrix $\boldsymbol{B}$ in equation (5) relate to those defining $\boldsymbol{Y}$ as

$$\frac{\partial^2 T_R}{\partial \sigma_i^s \partial \sigma_j^r} = \frac{\partial g_k^s}{\partial \sigma_i^s}\frac{\partial^2 T_R}{\partial g_k^s \partial g_l^r}\frac{\partial g_l^r}{\partial \sigma_j^r} , \qquad i,j=1,2; \quad k,l=1,2,3 . \tag{24}$$

In equations (23) and (24), we have introduced the coordinate transformations $(\partial q_i^{s,r}/\partial g_j^{s,r})$ and $(\partial \sigma_i^{s,r}/\partial g_j^{s,r})$ with $i = 1,2$ and $j = 1,2,3$ from $\vec{q}^{s,r}$ and $\vec{\sigma}^{s,r}$, respectively, to $\hat{\boldsymbol{g}}^{s,r}$.

Introducing the slowness vector components in the group-velocity centered coordinate systems at the source and receiver,

$$P_k^{s,r} = \frac{\partial T^{s,r}}{\partial g_k^{s,r}} \,, \tag{25}$$

we note that

$$\frac{\partial^2 T_R}{\partial g_k^s \partial g_3^r} = \frac{\partial^2 T^s}{\partial g_k^s \partial g_3^r} = \frac{\partial P_k^s}{\partial g_3^r} = 0 \tag{26}$$

and

$$\frac{\partial^2 T_R}{\partial g_3^s \partial g_l^r} = \frac{\partial^2 T^r}{\partial g_3^s \partial g_l^r} = \frac{\partial P_l^r}{\partial g_3^s} = 0 \,. \tag{27}$$

These derivatives vanish, because the slowness vector components P_k^s at the source $\boldsymbol{x}^s$ are independent of a perturbation of the receiver $\boldsymbol{x}^r$ along g_3^r, that is, *along the direction of the reflected ray.* Correspondingly, the P_k^r at $\boldsymbol{x}^r$ are independent of a perturbation of $\boldsymbol{x}^s$ along g_3^s. Therefore, the above relationships (23) and (24) remain valid with k and l only assuming the values 1 and 2. See section A.5 for a discussion of this crucial fact.

We may, thus, recast equations (23) and (24) into matrix form as

$$\boldsymbol{Q}_2^{-1} = \boldsymbol{\Gamma}^T(\boldsymbol{x}^s)\boldsymbol{Y}^{-1}\boldsymbol{\Gamma}(\boldsymbol{x}^r) \tag{28}$$

and

$$\boldsymbol{B}^{-1} = \boldsymbol{\Lambda}^T(\boldsymbol{x}^s)\boldsymbol{Y}^{-1}\boldsymbol{\Lambda}(\boldsymbol{x}^r) \,, \tag{29}$$

respectively. Here, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ are the upper left 2×2 submatrices of the full 3×3 transformation matrices $(\partial g_i^{s,r}/\partial q_j^{s,r})$ and $(\partial g_i^{s,r}/\partial \sigma_j^{s,r})$, respectively. All of these are general 3-D rotation matrices that can be decomposed into three elementary rotations, being one around the 3-axis, a second one around the resulting 2-axis, and a third one around the new 3-axis. Therefore, their upper left 2×2 submatrices can be decomposed into three elementary matrices, being two rotation matrices and a projection matrix, namely

$$\boldsymbol{\Gamma} = \begin{pmatrix} \cos\gamma_q & \sin\gamma_q \\ -\sin\gamma_q & \cos\gamma_q \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi_q & \sin\phi_q \\ -\sin\phi_q & \cos\phi_q \end{pmatrix} \tag{30}$$

and

$$\boldsymbol{\Lambda} = \begin{pmatrix} \cos\gamma_\sigma & \sin\gamma_\sigma \\ -\sin\gamma_\sigma & \cos\gamma_\sigma \end{pmatrix} \begin{pmatrix} \cos\chi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi_\sigma & \sin\phi_\sigma \\ -\sin\phi_\sigma & \cos\phi_\sigma \end{pmatrix} \,. \tag{31}$$

Here, $\phi_{q,\sigma}$ and $\gamma_{q,\sigma}$ denote the in-plane rotation angles around the old and new 3-axes, respectively. Also, α denotes the angle between the group velocity and the interface normal, and χ denotes the angle between the group and phase velocities. Both formulas (30) and (31) hold, correspondingly, at the ray's initial and end points, that is, at source and receiver.

Eliminating the auxiliary matrix $\boldsymbol{Y}$ from equations (28) and (29), we obtain the relationship

$$\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{\Gamma}^{-1}(\boldsymbol{x}^r)\boldsymbol{\Lambda}(\boldsymbol{x}^r)\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s)\boldsymbol{\Lambda}^T(\boldsymbol{x}^s)\boldsymbol{\Gamma}^{-T}(\boldsymbol{x}^s) \; . \tag{32}$$

Setting

$$\boldsymbol{G} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma} \tag{33}$$

and substituting it into equation (32) establishes the second claim (11).

Corresponding relationships to formula (32) hold for the matrices $\boldsymbol{B}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)$ and $\boldsymbol{B}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}})$ of the ray segments from the source to the reflection point and from there to the receiver. These are

$$\boldsymbol{B}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = \boldsymbol{\Gamma}_s^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{\Lambda}_s(\tilde{\boldsymbol{x}})\boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)\boldsymbol{\Lambda}^T(\boldsymbol{x}^s)\boldsymbol{\Gamma}^{-T}(\boldsymbol{x}^s) \tag{34}$$

and

$$\boldsymbol{B}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) = \boldsymbol{\Gamma}^{-1}(\boldsymbol{x}^r)\boldsymbol{\Lambda}(\boldsymbol{x}^r)\boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}})\boldsymbol{\Lambda}_r^T(\tilde{\boldsymbol{x}})\boldsymbol{\Gamma}_r^{-T}(\tilde{\boldsymbol{x}}) \; , \tag{35}$$

where $\boldsymbol{\Gamma}_{s,r}$ and $\boldsymbol{\Lambda}_{s,r}$ are the corresponding transformation matrices for the source and receiver rays, respectively, at the reflection point $\tilde{\boldsymbol{x}}$. They are given by equivalent equations to (30) and (31) for the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$.

Substitution of equations (32) to (35) into the decomposition formula (9) leads to

$$\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}})\boldsymbol{\Lambda}_r^T(\tilde{\boldsymbol{x}})\boldsymbol{\Gamma}_r^{-T}(\tilde{\boldsymbol{x}})\boldsymbol{H}(\tilde{\boldsymbol{x}})\boldsymbol{\Gamma}_s^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{\Lambda}_s(\tilde{\boldsymbol{x}})\boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \; . \tag{36}$$

We now take determinants of both sides of this equation. From the expressions for $\boldsymbol{\Gamma}_{s,r}$ and $\boldsymbol{\Lambda}_{s,r}$ that correspond to equations (30) and (31), respectively, we recognize that

$$\det \boldsymbol{\Gamma}_{s,r} = \cos\alpha^{s,r} \quad \text{and} \quad \det \boldsymbol{\Lambda}_{s,r} = \cos\chi^{s,r} \; . \tag{37}$$

Here $\alpha^{s,r}$ are the angles between the group velocity of the source and receiver rays and the surface normal at $\tilde{\boldsymbol{x}}$ and $\chi^{s,r}$ are the angles between the group and phase velocities of these rays at the same point. In this way, we finally obtain the desired formula (8).

Why the group-velocity centered coordinates?

Intuitively, one might be tempted to connect the second-derivative matrices defining $\boldsymbol{B}$ and $\boldsymbol{Q}_2$ by a direct transformation from ray-centered to local Cartesian coordinates. This would read, parallel to equations (23) and (24),

$$\frac{\partial^2 T_R}{\partial\sigma_i^s \partial\sigma_j^r} = \frac{\partial q_k^s}{\partial\sigma_i^s}\frac{\partial^2 T_R}{\partial q_k^s \partial q_l^r}\frac{\partial q_l^r}{\partial\sigma_j^r} \; , \qquad i,j = 1,2, \;\; k,l = 1,2,3 \; . \tag{38}$$

However, the second derivatives of the traveltime with respect to the 3-axes of the ray-centered coordinate systems do not vanish since these derivatives involve perturbations of the ray's end points in directions other than along the ray, that is,

$$\frac{\partial^2 T_R}{\partial q_k^s \partial q_3^r} \neq 0 \tag{39}$$

and

$$\frac{\partial^2 T_R}{\partial q_3^s \partial q_l^r} \neq 0 \ . \tag{40}$$

Therefore, equation (38) *does not* reduce to a 2×2 matrix relationship as is the case for equations (23) and (24). This reduction only occurs in isotropic media where $\hat{\boldsymbol{q}} = \hat{\boldsymbol{g}}$. This explains why the transformation from ray-centered to local Cartesian coordinates can be done in one step in isotropic media, but has to be done in two steps in anisotropic media, involving the group-velocity centered coordinate system as an intermediate step.

CONCLUSIONS

In this paper, we have shown how the decomposition formula for the geometrical-spreading factor (Tygel et al., 1994; Ursin and Tygel, 1997) generalizes to anisotropic media. The decomposition can, of course, be cascaded in the same way as shown in isotropic media by Hubral et al. (1995). The anisotropic decomposition formula has also been used to show that the Kirchhoff approximation (Bleistein, 1984) can be extended to anisotropic media as intuitively expected (Schleicher et al., 1999).

However, this is a result not only crucial for forward modeling purposes but it will also allow to set up the correct anisotropic counterparts to the weights for true-amplitude Kirchhoff migration (Schleicher et al., 1993).

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PUBLICATIONS

A paper containing these results has been submitted to Wave Motion.