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Analysis of an acoustic wave equation for cylinder symmetric media (2.5D)

R.S. Portugal, L.T. Santos, and J. Schleicher[1]

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ABSTRACT

The description of 3-D wave propagation in a 2-D medium (2.5-D situation) by means of a simple 2.5-D wave equation enables finite-difference (FD) reference solutions in 3-D with the cost of 2-D modeling. A comparison to alternative approximations in homogeneous and vertical-gradient media shows that the accuracy of a previously suggested approximate 2.5-D wave equation (Liner's equation) depends on how well the rms velocity can be approximated by a constant velocity. Kinematically, Liner's equation works very well up to very high velocity gradients. For an increasing velocity, amplitudes obtained by Liner's equation systematically underestimate true 3-D amplitudes, and for a decreasing velocity, they overestimate the true amplitudes. In conclusion, Liner's equation is a worthwhile alternative to other existing 2.5-D modeling schemes.

INTRODUCTION

In this work, we study three-dimensional (3-D) acoustic wave propagation in a vertical plane representing a medium the properties of which do not depend on the orthogonal direction to the plane.

This problem arises from the need to describe seismic data from the standard acquisition geometry: the two-dimensional (2-D) seismic line. Any modeling or inversion method based on such data must rely on some hypothesis permitting this geometric restriction. The simplest one is to consider the seismic wave propagation to occur only in the vertical plane containing the seismic line. This hypothesis implies the assumption that the medium parameters do not change in the orthogonal direction to the plane. In practice, of course, this means that the medium variations in the out-of-plane direction

[1] **email:** portugal@ime.unicamp.br

are so small that the resulting effects may be neglected. Such a medium is, in effect, a 2-D medium or, from the 3-D point of view, this medium has cylindrical symmetry.

The main goal of this paper is to study the wave propagation in this vertical plane which contains the seismic line and to compare three different approaches to describe this wave propagation.

Due to the high cost of 3-D modeling, there is a major interest in making use of the medium symmetry and applying a 2-D modeling scheme to describe the situation. However, as is well-known (Bleistein, 1986), modeling by the 2-D wave equation is not sufficient to describe the true 3-D wave propagation, although the traveltimes of the waves propagating within the plane of consideration can be modeled correctly. The reason is that the geometrical spreading of a seismic wave generated by a point source is a 3-D quantity that cannot be modeled by the 2-D wave equation. Therefore, as the medium and the kinematics of the problem are 2-D but the dynamics are 3-D, the problem is generally referred to as 2.5-D (Bleistein, 1986).

Because of the problems described above, 3-D modeling seems unavoidable for the correct description of 2.5-D wave propagation. However, several approaches exist in the literature to overcome these difficulties. On the one hand, 3-D methods can be adapted to the 2.5-D problem, i.e., they can be sped up making use of the medium symmetry analytically. Examples are ray theory and Kirchhoff modeling as described by Bleistein (1986). On the other hand, 2-D methods can be corrected for the wrong amplitude treatment of 3-D waves, as, e.g., the amplitude correction of the 2-D finite-difference (FD) solution also suggested by Bleistein (1986).

There is another 2-D concept which tries to overcome the difficulty of the 2.5-D problem. It is the concept of a partial differential equation called the 2.5-dimensional wave equation. Such an equation would correctly model the in-plane wave propagation, taking advantage of the kinematic part of 2-D propagation (which is correct) but treating the amplitudes as required in 3-D. From this ambivalence (2-D traveltimes and 3-D amplitudes) results the attribute 2.5-D.

One can precisely define the 2.5-D wave equation as a partial differential equation with the following properties:

- The 2.5-D wave equation is in fact 2-D, i.e., it depends on two spatial (x, z) and one temporal (t) coordinates;
- The equation has to simulate 3-D in-plane waves, i.e., waves propagating exclusively within the plane vertically below the receivers, but with a 3-D amplitude behavior.

In short, we may say

"A 2.5-D wave equation simulates 3-D wave propagation with the cost of 2-D modeling."

A 2.5-dimensional wave equation has the following conceptual advantage over the above other methods: It is not necessary to consider elementary events separately. Reflected waves, refracted waves, diffractions, multiples, among other events are all considered automatically in the 2.5-D simulation. The contrary is true in all other 2.5-D methods mentioned above, where one must define which events are to be modeled.

Note that the 2.5-D wave equation describes wave propagation correctly in the 2.5-D situation only. This equation is not correct in any plane in space but only within the symmetry plane.

To take advantage of the symmetry of the 2.5-D problem, Liner (1991) devised an approximate 2.5-D wave equation suitable to the finite-difference (FD) method. For the derivation of his equation, Liner (1991) relies on two basic assumptions. He assumes the medium to be homogeneous in the close vicinity of the source and he assumes the source pulse to be short. Furthermore, he conjectures that the validity of the obtained equation can be extrapolated to the region farther from the source, where the medium is no longer homogeneous.

Williamson and Pratt (1995) show that Liner's equation is mathematically not correct. In this paper, we study its numerical validity, i.e., whether its modeling results, although incorrect, remain a good approximation for the 3-D waves in a 2-D medium. In other words, it is our aim to verify whether 2.5-D modeling by Liner's equation is still a worthwhile alternative to other 2.5-D methods. A similar investigation was carried out by Bording and Liner (1993). Here, we extend their results to media with a constant velocity gradient.

METHODOLOGY

The Green's function, $G_{2.5}(x,z,t)$, is the in-plane solution of the 3-D wave equation for a source term in form of a delta pulse. Due to the nature of the problem,

$$G_{2.5}(x,z,t) \;=\; G_3(x,y=0,z,t), \tag{1}$$

where $G_3(x,y,z,t)$ is the Green's function of the 3-D wave equation. Unfortunately, there exists no 2-D differential equation with a source term in form of a delta pulse with $G_{2.5}(x,z,t)$ as its solution, i.e., $G_{2.5}(x,z,t)$ is not the Green's function to any possible 2.5-D wave equation.

Liner's approach to find an approximate 2.5-D wave equation (Liner, 1991) was simple and straightforward. He searched for a differential operator which, applied to the 3-D in-plane Green's function for a homogeneous medium, $G_{2.5}(x,z,t)$, yields zero. In other words, he sought the 2-D differential operator that propagates the wavefield described by $G_{2.5}(x,z,t)$ within a 2-D medium correctly accounting for the 3-D amplitude effects in the region away from the sources. He avoided the difficulties with the source term by starting the propagation at a small time $t_1 > 0$ with the homogeneous 3-D Green's function defining the initial conditions at this time t_1.

Liner expected the searched-for 2.5-D wave equation to be somehow related to the 2-D wave equation. According to this idea, he simply applied the 2-D wave equation to $G_{2.5}(x,z,t)$ and moved the resulting terms on the right-hand side back to the left of the equation. Under the additional assumptions that the pulse to be propagated has a short duration and that the medium velocity is constant near the source, he arrived at

$$\left[\frac{1}{c^2}\left(\partial_{tt}+\frac{1}{t}\partial_t+\frac{1}{t^2}\right)-\partial_{xx}-\partial_{zz}\right]G_{2.5}(x,z,t) \;=\; 0, \tag{2}$$

where $c = c(x,z)$.

For further analysis of Liner's equation (2), we follow Stockwell (1995) to study its approximate behavior using the ray-theory ansatz

$$G_{2.5}(x,z,t) \;=\; A_{\rm L}(x,z)\exp[\,iw(t-\tau(x,z))\,]. \tag{3}$$

Here, $\tau(x,z)$ is the traveltime from the source, in the center $(0,0)$ of the initial wavefield at t_1, to point (x,z). Also, $A_{\rm L}(x,z)$ is the wavefield amplitude at that point. Inserting ansatz (3) into equation (2) yields Liner's eikonal equation

$$\|\nabla\tau\|^2-\frac{1}{c^2} \;=\; 0 \tag{4}$$

and Liner's transport equation

$$2\,\nabla A_{\rm L}\cdot\nabla\tau \;+\; A_{\rm L}\,\Delta\tau \;+\; \frac{A_{\rm L}}{c^2\,\tau} \;=\; 0. \tag{5}$$

In fact the former is simply the 2-D eikonal equation, proving the traveltimes determined by Liner's equation to be correct. However, equation (5) is different from the correct 2.5-D transport equation,

$$2\,\nabla A_{2.5}\cdot\nabla\tau + A_{2.5}\,\Delta\tau + \frac{A_{2.5}}{\sigma} \;=\; 0, \tag{6}$$

which can be obtained from the 3-D transport equation by carrying out the y-derivatives involved. Moreover, σ is given by

$$\sigma \;=\; \int_0^\tau c^2\,d\tau', \tag{7}$$

integrated along the Fermat ray. Note that σ can be interpreted as an rms velocity measured along the ray.

The difference between equations (5) and (6) leads to the following relation between the corresponding amplitudes

$$A_{2.5} \;=\; \alpha\, A_{\mathrm{L}}, \tag{8}$$

where $\alpha = \sqrt{c_0^2 \tau / \sigma}$ (Stockwell, 1995). This equation shows how to correct Liner's amplitude to obtain the true 2.5-D amplitude. For a homogeneous medium, $\alpha = 1$, showing Liner's amplitude to be correct in the zero-order ray-theoretical approximation. For an inhomogeneous medium, Liner's amplitude A_{L} is seen to be a good approximation to the true amplitude $A_{2.5}$ as long as the integral expression (7) is well approximated by

$$\sigma \;=\; \int_0^\tau c^2\, d\tau' \;\approx\; \int_0^\tau c_0^2\, d\tau' \;=\; c_0^2\, \tau. \tag{9}$$

Constant velocity gradient

To further study the amplitude error of Liner's equation, we consider the case of a constant velocity gradient in depth, i.e.,

$$c \;=\; c(z) \;=\; c_0 \;+\; a\, z, \tag{10}$$

with c_0 and a constants. In this situation, an analytic formula for the amplitude correction factor α can be obtained, namely

$$\alpha \;=\; c_0\, p \left\{ \ln \left[\frac{(1+\cos\theta)\, c}{c_0 + c_0\sqrt{1-p^2\, c^2}} \right] \; \frac{1}{\cos\theta - \sqrt{1-p^2\, c^2}} \right\}^{1/2}, \tag{11}$$

where $p = \sin\theta / c_0$ is the ray parameter and θ is the departure angle of the ray.

In the following section, the error of Liner's amplitude predicted by this formula is compared to numerical results.

NUMERICAL ANALYSIS

In this section, we numerically compare three methods that describe the wave propagation in the 2.5-D situation. The first one is the ray theory solution, the second one is the FD solution of a 2-D wave equation with an appropriate amplitude correction (Bleistein, 1986), and the third one is the FD solution of Liner's equation (2). Of several synthetic examples presented by Portugal (1998), here we discuss in detail the results obtained for a model with a horizontal planar reflector at depth of 400 m below an acoustic medium with a constant vertical velocity gradient of 1 km/s per kilometer of depth.

We simulated a common-shot experiment with 26 receivers at every 25 m, the first one being located at zero offset and the last one at an offset of 625 m. The source pulse was a Ricker wavelet with a dominant frequency of about 28 Hz. Figure 1 shows the pulse of the primary reflected event in the seismic trace recorded by a geophone at an

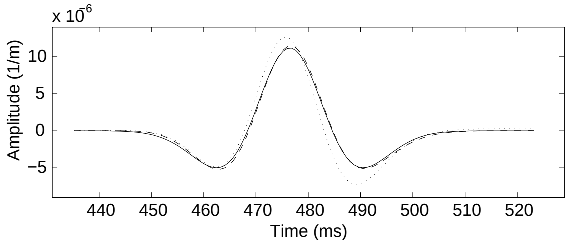

Figure 1: Reflected wave seismic trace recorded at 300 m offset. Modeling by ray theory (continuous line), 2-D FD with amplitude correction (dashed line) and Liner's equation (dotted line)

offset of 300 m from the source, as modeled by ray theory, 2-D FD with amplitude correction, and FD solution of Liner's equation.

The numerical analysis was divided into two parts, considering separately the modeling of traveltimes and amplitudes. In all cases, the traveltime analysis showed satisfactory coincidence between all three methods. All errors remained below 0.5%. This result is in accordance with the expectation based on the theoretical observation that both eikonal equations, namely Liner's eikonal equation and the true 2.5-D eikonal equation, are the same.

The dynamic comparison was restricted to amplitude peaks letting aside possible differences between the modeled pulse forms. As can be observed from Figure 1, Liner's equation produces some minor distortion in the wave form as compared to both other methods, due to the violation of the short-pulse assumption.

Concerning peak amplitudes, Liner's equation yields good results in numerical experiments for relatively simple models, in particular for constant velocity. These results, although obtained for small models, can be without doubt extended to more realistic situations since the theoretical analysis shows that for this case Liner's amplitudes are correct in the sense of zero-order ray approximation.

For our constant-gradient model, where the assumption of Liner (1991) are no longer satisfied, greater errors can be expected. This can be seen in Figure 1, which shows very good coincidence between the amplitudes of ray theory and amplitude corrected 2-D FD, but a slight deviation of those from Liner's equation. This is in accordance with the theoretical result given by equation (11).

A more detailed analysis of peak amplitudes as a function of offset is depicted in Figure 2. Figure 2a shows the peak amplitudes themselves and Figure 2b shows the

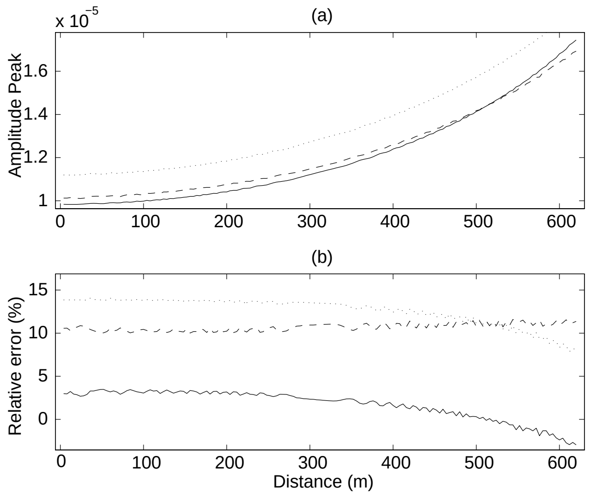

Figure 2: (a) Peak amplitudes recorded at offset geophones. Modeling by ray theory (continuous line), 2-D FD with amplitude correction (dashed line) and Liner's equation (dotted line). (b) relative error between: 2-D FD with amplitude correction and ray theory (continuous line), Liner's equation and 2-D FD with amplitude correction (dashed line) and Liner's equation and ray theory (dotted line).

corresponding relative errors between each pair of methods. From Figure 2a it can be observed that the amplitudes increase with offset. The amplitudes obtained from Liner's equation are systematically greater than the amplitudes obtained by the other two methods. We also note that ray theory and 2-D FD with amplitude correction amplitudes are in much better coincidence than Liner's amplitude with any of them.

This fact can be better observed in Figure 2b. The relative error between Liner's amplitudes and those of the others two methods remains within the 10–15% range. The relative error between ray theory and 2-D FD with amplitude correction is much smaller, never exceeding 5%.

To extrapolate these results to other constant-gradient models, we have employed the analytic expression for $\alpha = A_{2.5}/A_{\rm L}$ given in equation (11).

Figure 3 shows the prediction of the relative error of the modeling of a direct or a reflected vertical wave in a medium with a constant vertical velocity gradient, as determined using formula (11), i.e., $(\alpha - 1)/\alpha$. We see that the errors for a negative and positive velocity gradient are on opposite sides of the axis. This seems to suggest that the error of a reflected wave should have canceling contribution from the upgoing and downgoing ray. This is, however, not true. In fact, the error of the wave reflected at a certain depth z and recorded at $z = 0$ is identical to the error of the direct wave observed at depth z (Portugal, 1998).

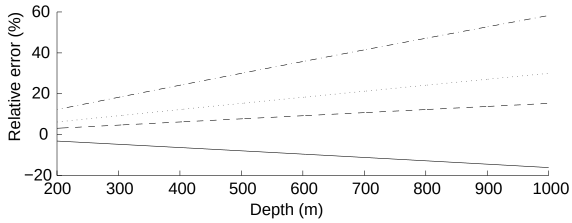

Figure 3: Theoretical prediction of the relative error of Liner's amplitude for a vertical wave in a medium with a constant vertical velocity gradient. The same error graphs apply to the direct wave recorded at depth z and the wave reflected at depth z and recorded at $z = 0$. Continuous line: gradient of -0.5 s^{-1}; dashed line: gradient of 0.5 s^{-1}; dotted line: gradient of 1 s^{-1}; dash-dotted line: gradient of 2 s^{-1}.

The result of an error of about 14% as obtained by our synthetic example (see Figure 2) for a reflector at a depth of 400 m in a medium with a vertical velocity gradient of 1 s^{-1} falls exactly on the corresponding dotted line. We see that the theoretical prediction of the error is quite exact. This confirms that α is a good measure for the actual amplitude errors of Liner's equation. ¿From Figure 3, we observe that Liner's amplitudes get fairly wrong for strong velocity gradients and greater depths.

As a final remark, note that in all experiments we have carried out in this connection, the difference between ray theory and Liner's equation was greater than the one between the latter equation and the amplitude-corrected 2-D wave equation. This might be due to the fact the both equations are solved with an identical FD scheme, thus giving rise to the same types of numerical errors.

CONCLUSION

In conclusion, Liner's equation is a worthwhile alternative for the modeling of 3-D wave propagation in 2.5-D situations with weak velocity variations. Although slightly erroneous in the resulting amplitudes, it remains a good approximation for models not too complex. Its amplitudes are an order of magnitude better approximations to the true 2.5-D amplitudes than those of the 2-D wave equation obtained with the same computational cost. If a good approximation for the ray parameter σ can be obtained, the amplitudes of Liner's equation can be further corrected, without the need for an additional pulse-form correction as is necessary for the amplitude correction of the 2-D wave equation. However, for a not so small velocity gradient and greater depth, the amplitudes modeled by Liner's equation are not reliable. Future research is suggested on possible improvements of Liner's equation for inhomogeneous media.

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