Multifocus Moveout Revisted: Derivations and Alternative Expressions

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ABSTRACT

The multifocus moveout of Gelchinsky and coworkers is a powerful tool for stacking multiview data in arbitrary configurations. Based on general ray theoretical assumptions and on attractively simple geometrical considerations, the multifocus moveout is designed to express the traveltimes of neighboring rays arbitrarily located around a fixed central, primary reflected or even diffracted, ray. In this work, the basic derivations and results concerning the multifocus approach are reviewed. A higher-order multifocus moveout expression that generalizes the corresponding one of Gelchinsky is obtained from slight modifications of the original derivation. An alternative form of the obtained multifocus expression that is best suited for numerical implementation is also provided.

INTRODUCTION

Accurate and reliable traveltime moveout expressions are of prime importance in seismic processing and imaging because of their use in producing stacked sections. The most famous moveout expression is the normal and dip moveout (NMO/DMO) designed to describe the traveltime of primary reflections of common-midpoint (CMP) data. In many seismically relevant situations, the NMO/DMO process has been able to produce stacked sections with significant reduction of noise, also attenuating multiples and other undesirable events. Although CMP stacking under NMO/DMO is a routine step in practically all seismic processing sequences in the oil industry, also a number of shortcomings of the method have been recognized. Being designed for gently dipping reflectors and small lateral velocity variations in the overburden, and moreover, for not too large offsets, the NMO/DMO moveout expression are no longer accurate when these conditions are severely violated. A second shortcoming is its dependence of the CMP configuration. In modern acquisition surveys, CMP data represents a fraction of the acquired data. As a consequence, moveout expressions that use arbitrary

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locations of source and receiver pairs around a fixed central point (that may even be a CMP point) are able to make a much better use of the available data and to profit from the significantly greater redundancy that is offered.

Several traveltime moveout formulas already exist in the literature that describe traveltimes along neighboring rays of a fixed central normal, zero-offset ray, with arbitrary locations of the source and receiver around the central point. These are the classical parabolic/hyperbolic moveouts (see, e.g., Ursin, 1982; Cervený, 1985; Schleicher et al., 1993), the optical stack moveout of de Bazelaire (1988), the multifocus moveout of Gelchinsky (1988) and the recent common reflection surface (CRS) moveout of Höcht (1998). All the above traveltime moveout formulas are coincident in the second-order approximation of source and receiver offsets. An actual and objective comparison between them is not an easy task and remains a challenging problem and some research in this topic would be certainly welcome. With the exception of the classical hyperbolic and parabolic moveouts, all the other formulas are, up to now, two-dimensional, which means that sources and receivers are located on a single seismic line and the medium does not vary in the out-of-plane direction.

In this note, we concentrate on the geometric appealing multifocus moveout of Gelchinsky and coworkers. We feel it has not attracted the attention it deserves in the seismic literature and, perhaps, a great deal of its potential has not being sufficiently exploited. By reviewing the multifocus original derivations and results, as provided in several publications since the first presentation of Gelchinsky (1988), and summarized in Gelchinsky et al. (1997), we introduce a new expression of Gelchinsky's multifocus parameter that is not only slightly more general but also implementationally more stable. Substitution into the original multifocus moveout formula leads to a higher-order approximation expression in terms of source and receiver offsets. The new parameter reduces to its previous counterpart by natural approximations. As a second contribution of the present analysis, we introduced a modification in the definition of Gelchinsky's asymmetry parameter, so as to have it dimensionless. Moreover, working with the reciprocal of the newly introduced multifocus parameter led to an alternative, mathematically equivalent moveout formula, which is more amenable for numerical implementation.

GELCHINSKY'S MULTIFOCUS MOVEOUT

We assume that the actual subsurface, although unknown, can be described by a 2-D laterally inhomogeneous isotropic layered earth model. In this model, we further assume that the kinematics of body waves is well described by zero-order ray theory (see, e.g., Cervený, 1985). We use Cartesian coordinates \((x, z)\) and suppose that a dense multi-coverage seismic experiment has been carried out on a single seismic line along the \(x\)-axis. This implies that each point of the seismic line is surrounded by a set
of shot-receiver pairs (within a certain range of offsets). The discreteness of real-world data may require trace interpolation to replace missing traces.

Referring to Figure 1, we consider a fixed target reflector $\Sigma$ in depth, as well as a fixed central point $X_0$ on the seismic line, considered to be the location of a coincident source-receiver pair $S_0 = G_0 = X_0$. Also shown in Figure 1 is the two-way normal, zero-offset reflection ray, called from now on the central ray. It hits the reflector at point $R_0$, known as the normal-incident-point (NIP). Figure 1 finally shows a pair of source and receiver points $(S, G)$ together with its corresponding primary reflected ray $SRG$, relative to the target reflector $\Sigma$. The source and receiver pair $(S, G)$ will be consider a generical description of all source-receiver pairs in the vicinity of the central point. We note incidently that the central ray, as well as the reflection ray $SRG$ focus at point $P$ in depth. This fact will be of importance later on. We use the horizontal coordinates $x_0$, $x_S$ and $x_G$ to specify the location of the central point $X_0$, the source $S$ and the receiver $G$, respectively.

The relative distances from a given source-receiver pair $(S, G)$ to the fixed central point $X_0$

$$\Delta x_S = x_S - x_0 \quad \text{and} \quad \Delta x_G = x_G - x_0,$$

are called the source and receiver offsets, respectively.

Referring again to Figure 1, it is our aim to find an approximation of the travel-time of the reflection ray $SRG$ in the vicinity of the central, zero-offset reflection ray $X_0 R_0 X_0$. We assume that the traveltime of latter, as well as the medium velocity at the central point are given by the quantities $T_0$ and $v_0$, respectively. Suppose, as depicted in Figure 1, that the two rays $SRG$ and $X_0 R_0 X_0$ cross at the unknown point $P$. Without loss of generality, we assume $P$ to be on the source ray segment $SR$. The multifocus approach makes use of a hypothetical wave that originates at point $P$. This hypothetical wave is depicted in Figure 1 by two of its wavefronts. One, denoted by $\Sigma_S$, contains $X_0$ and has traveled up from $P$ to the source point $S$. The other, denoted by $\Sigma_G$, also contains $X_0$ and has traveled from $P$ down to the reflector $\Sigma$ and from there to the receiver point $G$. We denote the curvatures of these two wavefronts by $K_S$ and $K_G$, respectively. Note that a true wave originating at $S$ with an initial curvature of $-K_S$ focuses at $P$ and emerges at $G$ with curvature $K_G$. We will refer to this wave as the focusing wave.

By construction, the traveltime of the focusing wave from one wavefront $\Sigma_S$ to the other, $\Sigma_G$, is the given zero-offset traveltime $T_0$. As a consequence, we can express the traveltime for the ray $SRG$ in the form

$$T = T_0 + \Delta T_S + \Delta T_G,$$

in which $\Delta T_S$ and $\Delta T_G$ are the multifocus source and receiver moveouts. Let the medium velocity in the neighborhood of the central point $X_0$ be constant and denoted by $v_0$. Upon the assumption that the wavefronts $\Sigma_S$ and $\Sigma_G$ can be approximated by
circles with radii \( R_S = 1/K_S \) and \( R_G = 1/K_G \) and centers \( C_S \) and \( C_G \), respectively, it can be shown by simple geometrical considerations (see Figure 2) that the above multifocus moveouts can be easily determined as follows.

In Figure 2, the situation is explained at the source point \( S \). An analogous construction is valid at the receiver point \( G \). The multifocus moveout \( \Delta T_S \) is the travel-time from \( S \) to \( S' \), assuming the segment \( SS' \) to be a straight line. Within the triangle \( SCS'X_0 \) we have by the law of cosines that

\[
SC^2 = SX_0^2 + CSX_0^2 - 2SX_0 \cdot CSX_0 \cos\left(\frac{\pi}{2} + \beta_0\right) .
\]

Identifying \( SX_0 = \Delta x_S \), \( CSX_0 = CSX_0 = R_S = 1/K_S \), and \( SCX_0 = R_S + SS' \), solving equation (3) for \( SS' \), and dividing by \( v_0 \), we arrive at

\[
\Delta T_S = \frac{1}{v_0 K_S} \left[ \sqrt{1 + 2K_S \sin \beta_0 \Delta x_S + (K_S \Delta x_S)^2} - 1 \right] ,
\]

where we have chosen the sign of the square root according to the physical condition that \( \Delta T_S \) has to be positive for positive curvature \( K_S \). The same formula with all indices \( S \) changed to \( G \) holds for \( \Delta T_G \).

**The two fundamental eigenwaves**

As shown by Gelchinsky et al. (1997) using basic dynamic ray tracing arguments (see Appendix), the curvatures \( K_S = 1/R_S \) and \( K_G = 1/R_G \) of the down- and upgoing wavefronts of the hypothetical focusing wave are not independent. In our notation, they satisfy the relationships

\[
\frac{K_N - K_{NIP}}{1 - \Gamma} \quad \text{and} \quad \frac{K_{NIP} + \Gamma K_N}{1 + \Gamma} .
\]

In the above formula, \( K_N \) and \( K_{NIP} \) are the curvatures of the classical normal (N) wave and the normal-incidence-point (NIP) wave introduced by Hubral (1983) in connection with true-amplitude migration. Moreover, \( \Gamma \) is a modified version of the focusing parameter of Gelchinsky et al. (1997). It is defined as the reciprocal of the original focusing parameter \( \gamma \) introduced by Gelchinsky, i.e., \( \Gamma = 1/\gamma \). The involved \( N \) and \( NIP \) waves are two hypothetical waves defined as follows: (a) the \( N \) wave starts as a wavefront that coincides with the target reflector \( \Sigma \), propagates upwards with half the medium velocity and arrives at the central point \( X_0 \) at time \( T_0 \) and (b) the \( NIP \) wave starts at the target reflector \( \Sigma \) as a point source at the \( NIP \) point \( R_0 \), propagates upwards with half the medium velocity and arrives at the central point \( X_0 \) also at time \( T_0 \). As is well known (Hubral, 1983) both these waves are eigenwaves in the following sense: if their wavefronts at \( X_0 \) propagate downwards, reflect at \( \Sigma \) and propagate upwards to the surface, their arriving wavefronts at the central point \( X_0 \) coincide with
their corresponding initial wavefronts. Moreover, the relative geometrical-spreading factors of the $N$ and $NIP$ waves are plus or minus unity at $X_0$, respectively.

The modified focusing parameter $\Gamma$ controls the location along the central ray (or along its continuation) of the focusing point $P$ which is determined by the central ray and the neighboring reflecting ray $SRG$. In other words, for neighboring reflecting rays, we have a one-to-one correspondence between the value of the focusing parameter and the location of the focusing point. This is the reason why the present approximation formulas are called multifocus. Let us see how this applies to the just discussed $N$ and $NIP$ waves. Up to second-order approximation of the traveltime with respect to the source and receiver offsets, the $N$ wave can be considered as a wave focusing at the center of curvature of the reflector, because neighboring rays to the central ray are also normal rays. This wave is described by setting the focusing parameter $\Gamma = \Gamma_N = \infty$. Substitution into equation (5) yields $K_S = K_G = K_N$ as expected. In the same approximation, the $NIP$ wave can be considered as wave focusing at the $NIP$ point $R_0$. The focusing parameter for this wave is $\Gamma = \Gamma_{NIP} = 0$. We find from equation (5), $K_S = K_G = K_{NIP}$, as required. With the introduction of the modified focusing parameter $\Gamma$, the physical interpretation for the $N$ and $NIP$ waves are much more appealing. For instance, one can directly observe from the above that for positive $\Gamma$, the focus point $P$ falls below the reflector – or, in other words, onto the upgoing ray segment $RG$ – and for negative $\Gamma$, $P$ is above the reflector or on segment $SR$.

For the relationship between the multifocus parameter $\gamma$ and any actual source and receiver offsets $\Delta x_S$ and $\Delta x_G$, Gelchinsky et al. (1997) obtained the approximation [see their eq. (17), here corrected for a wrong sign and a factor of 2]

$$\begin{align*}
\gamma &= \frac{\Delta x_G - \Delta x_S}{\Delta x_G + \Delta x_S - K_{NIP} \sin \beta_0 \Delta x_G \Delta x_S}.
\end{align*}$$

(6)

This formula is valid to second-order approximation in $\Delta x_S$ and $\Delta x_G$. Up to first-order in the source-receiver offsets, the simpler expression is obtained

$$\begin{align*}
\gamma &= \gamma_0 = \frac{\Delta x_G - \Delta x_S}{\Delta x_G + \Delta x_S}.
\end{align*}$$

(7)

The above formula has been also obtained by Tygel et al. (1997) by an independent method, but following the same multifocusing principles.

**ALTERNATIVE MULTIFOCUS EXPRESSIONS**

In this section, we introduce some alternative definitions and expressions that relate to the multifocus moveout. The main objective of the new formulas is to have them in a most accurate and useful form, especially for direct numerical implementation. The obtained results followed upon slight modifications of the derivations of the original multifocus expressions, as presented in Gelchinsky et al. (1997).
The modified focusing parameter

The alternative multifocus moveout expression to be presented below, will be given in terms of the modified focusing parameter, \( \Gamma = 1 \gamma \). As shown in the Appendix, we find for the modified focusing parameter, the expression

\[
\Gamma = \Gamma_0 + \frac{\Lambda}{2} (1 + \Lambda) (1 - \Gamma_0^2)
\]

in which \( \Gamma_0 \) is the reciprocal of the zero-order approximation of the original multifocus parameter (see equation (7))

\[
\Gamma_0 = \frac{1}{\gamma_0} = \frac{\Delta x_G + \Delta x_S}{\Delta x_G - \Delta x_S},
\]

and \( \Lambda \) is the modified asymmetry parameter given by

\[
\Lambda = \frac{1}{2} (\Delta x_G - \Delta x_S) K_{NIP} \sin \beta_0.
\]

The modified asymmetry parameter defined above is nothing else than a dimensionless counterpart of the original asymmetry parameter \( \alpha = K_{NIP} \sin \beta_0 \) introduced by Gelchinsky (1988) in the description of the Common Reflection Element (CRE) Method. The asymmetry parameter plays a significant role in the selection of a source-receiver gather for which all its corresponding reflection rays reflect on a single point. The modification of the asymmetry parameter deserves an explanation. In the derivation of the formula for the traveltime, we have to perform some Taylor expansions for small values of the asymmetry parameter. Therefore, it is necessary to have it dimensionless in order to have a well-defined meaning for “small”.

Reduction to previous formulas

We consider the approximation of the modified focusing parameter \( \Gamma \) when the modified asymmetry parameter \( \Lambda \) becomes small. Comparing formulas (7) and (6) with the new approximation (8), we readily recognize that the former ones are the zero-order (\( \Lambda \approx 0 \)) and first-order (\( \Lambda^2 \approx 0 \)) approximations, respectively, of the latter.

Proposed multifocus formula

To present the multifocus traveltime expression suitable for numerical implementation, we start by rewriting formula (4) for the multifocus moveout \( \Delta T_j \) as

\[
\Delta T_j = \frac{\Delta x_j}{v_0 k_j} \left[\sqrt{1 + 2 k_j \sin \beta_0 + k_j^2} - 1\right], \quad j = S, G,
\]
where we have introduced the *dimensionless curvatures*

\[ k_j = \Delta x_j K_j, \quad j = S, G. \]  

The proposed alternative, mathematically equivalent multifocus traveltime expression (compare with equations (2) and (4)) is

\[ T = T_0 + \frac{1}{v_0} \left[ M_S \Delta x_S + M_G \Delta x_G \right], \]  

where

\[ M_j = \frac{k_j + 2 \sin \beta_0}{1 + \sqrt{1 + k_j(k_j + 2 \sin \beta_0)}}, \]  

and

\[ k_j = \frac{\Delta x_G - \Delta x_S}{2 - \Lambda(1 + \Lambda)(\Gamma_0 \pm 1)} [\Gamma K_N \mp K_{N1P}], \]  

in which the upper sign holds for \( j = S \) and the lower one for \( j = G \). In zero-order approximation (\( \Lambda = 0 \)) the above expression (15) for \( k_j \) reduces to

\[ k_j = \frac{(\Delta x_G + \Delta x_S)}{2} K_N \mp \frac{(\Delta x_G - \Delta x_S)}{2} K_{N1P}. \]  

The motivation behind the above formulas is that in numerical computations dimensionless quantities and large positive denominators are welcome, in order to prevent for overflow or underflow problems. Moreover, for \( K_j \approx 0 \), the original multifocus formulas (2) to (4) are more prone to loss of significant digits, whereas in the new ones this problem is overcome.

**CONCLUSIONS**

We have taken a closer look at the derivations and expressions for the multifocus moveout as elaborated in the last ten years by Gelchinsky and his coworkers and summarized in Gelchinsky et al. (1997). The original derivations were reviewed so as to make them more accessible to a broader audience and to put them into best implementable form. In the process, we obtained a slightly more general multifocus moveout formulas that reduces to the original when approximations for small source and receiver offsets are taken. We also introduced some modifications in the original asymmetry and multifocus parameters of Gelchinsky with the aim of having the final formulas more amenable to numerical implementation. At this stage, we make no claims the alternative formulas being better approximations than the original ones. Which are better will be seen only after application of the results to concrete examples. In our opinion, the present study should contribute to a better understanding of the fundamental as well as geometrical appealing and attractive multifocus idea, which deserves a better recognition
in the seismic literature. As a final observation, we mention that a multifocus move-out formula in three-dimensions is still not available. This is a topic of undergoing research.
REFERENCES


Figure 1: Shown are the normal ray $X_0R_0X_0$ and a pair of source and receiver points $(S, G)$ together with its corresponding primary reflected ray $SRG$, relative to the target reflector $\Sigma$. The normal ray and the reflection ray focus at point $P$ in depth. Also depicted are two wavefronts of the focusing wave: one travels on its way down to the reflector ($\Sigma_S$) and another travels on its way up to the surface ($\Sigma_G$). The emergence angle of the normal ray is denoted by $\beta_0$. 
Figure 2: Geometrical construction of multifocus moveout $\Delta T_S$. For details see text.
APPENDIX A

Multifocus Parameter

In this Appendix we derive the expression (8) for the modified multifocus parameter \( \Gamma \) introduced in equations (5). This derivation follows closely the one provided by Gelchinsky et al. (1997) using the basic concepts of ray theory. For the terminology and results to be used below, the reader is referred to Cervený (1985).

Let us first show how equations (5) follow from standard ray-theoretical arguments. We start by considering a selected planar ray path in a two-dimensional isotropic model. We assume that the ray is parameterized by the arclength \( s \). Points in the vicinity of this central ray will be described in ray-centered coordinates \((s, q)\), in which \( q \) is the transversal coordinate along the ray. The dynamic description of this ray is provided by the scalar quantities \( P = P(s) \) and \( Q = Q(s) \) computed along the ray, which satisfy the dynamic ray tracing system

\[
\frac{d}{ds} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 & -v_{qq} \\ v & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}
\]

Here, \( v_{qq} = v_{qq}(s) \) denotes the second derivative of the medium velocity with respect to the transversal coordinate \( q \), evaluated at the point of the ray determined by \( s \). As well known, the quantity \( Q = Q(s) \) is the square of the point-source, relative geometrical spreading along the ray.

The central ray under consideration is the zero-offset primary reflection ray introduced in the text (see Figure 1). This ray starts and ends at the central point \( X_0 \). For definiteness, the (coincident) source and receiver points will be parametrized by \( s = 0 \) and \( s = \ell \), respectively. We have, of course, that \( v(0) = v(\ell) = v_0 \).

In terms of quantities \( P(s) \) and \( Q(s) \), the wavefront curvature \( K(s) \) along the ray can be expressed by

\[
K(s) = v(s) \frac{P(s)}{Q(s)}.
\]

From general properties of linear systems, the general solution of the dynamic ray system (A-1) can be written as a linear combination

\[
\begin{bmatrix} P \\ Q \end{bmatrix} = a \begin{bmatrix} P^1 \\ Q^1 \end{bmatrix} + b \begin{bmatrix} P^2 \\ Q^2 \end{bmatrix},
\]

(A-3)
of two independent, arbitrarily fixed solutions \((P^1, Q^1)\) and \((P^2, Q^2)\), \( a \) and \( b \) being constants. Each basic solution pair \((P^i(s), Q^i(s)) (i = 1, 2)\) defines an elementary
wave that propagates in the vicinity of the central ray. It turns out that, within the
validity of the paraxial ray theory, any elementary wave that propagates in the vicinity
of the central ray and is of the same type as the elementary wave propagating along
that ray is described by a solution pair \((P(s), Q(s))\) of the form (A-3). Using equation
(A-3) we can rewrite equation (A-2) for the curvature at each point of the ray as

\[
K(s) = \frac{q(s)K^1(s) + K^2(s)}{1 + q(s)},
\]

(A-4)

where

\[
K^i(s) = v(s)\frac{P^i(s)}{Q^i(s)}, \quad i = 1, 2 \quad \text{and} \quad q(s) = \frac{a}{b} \frac{Q^1(s)}{Q^2(s)}.
\]

(A-5)

Let us now consider the particular focusing wave that starts at \(S\) with wavefront
curvature \(K(0) = -K_S\) and emerges at \(G\) with wavefront curvature \(K(\ell) = K_G\) (see
Figure 1). From equation (A-4) we find

\[
K_S = -K(0) = -\frac{q(0)K^1(0) + K^2(0)}{1 + q(0)},
\]

(A-6)

and

\[
K_G = K(\ell) = \frac{q(\ell)K^1(\ell) + K^2(\ell)}{1 + q(\ell)},
\]

(A-7)

in which

\[
q(0) = \frac{a}{b} \frac{Q^1(0)}{Q^2(0)} \quad \text{and} \quad q(\ell) = \frac{a}{b} \frac{Q^1(\ell)}{Q^2(\ell)}.
\]

(A-8)

As natural choice for the basic solutions, we select the pairs \((P_N, Q_N)\) and \((P_{NIP}, Q_{NIP})\)
that correspond to the \(N\) and \(NIP\) eigenwaves introduced by Hubral (1983). These
very special elementary waves are characterized by the following two properties:

1. Both waves start and end at the central point \(X_0\), their final wavefronts being
   coincident with the respective initial ones. Because of the opposite direction of
   propagation at the initial and endpoints, each eigenwave has curvatures of equal
   modulus but opposite signs at the coincident source and receiver points;

2. The relative geometrical-spreading factors of the \(N\) and \(NIP\) waves at the end
   point \(X_0\) are plus or minus one, respectively.

For a more detailed description and application of the \(N\) and \(NIP\) eigenwaves, the
reader is referred to Hubral (1983). The above-described properties of the \(N\) and
\(NIP\) eigenwaves translate mathematically into the relationships

\[
K_N(\ell) = \frac{P_N(\ell)}{Q_N(\ell)} = -\frac{P_N(0)}{Q_N(0)} = -K_N(0),
\]

(A-9)
and
\[ \frac{Q_N(\ell)}{Q_N(0)} = 1 , \quad (A-10) \]
as well as
\[ K_{N1P}(\ell) = \frac{P_{N1P}(\ell)}{Q_{N1P}(\ell)} = - \frac{P_{N1P}(0)}{Q_{N1P}(0)} = - K_{N1P}(0) , \quad (A-11) \]
and
\[ \frac{Q_{N1P}(\ell)}{Q_{N1P}(0)} = -1 . \quad (A-12) \]

In accordance with equations (A-9) and (A-11), we introduce the notations
\[ K_N = K_N(\ell) = -K_N(0) \quad \text{and} \quad K_{N1P} = K_{N1P}(\ell) = -K_{N1P}(0) . \quad (A-13) \]

In accordance with equations (A-10) and (A-12) inserted into equation (A-8), we also introduce the modified focusing parameter,
\[ \Gamma = q(\ell) = \frac{a}{b} \frac{Q_N(\ell)}{Q_{N1P}(\ell)} = - \frac{a}{b} \frac{Q_N(0)}{Q_{N1P}(0)} = - q(0) . \quad (A-14) \]

Substituting notations (A-13) and (A-14) into the curvature equations (A-6) and (A-7), we find the following expressions for the source and receiver wavefront curvatures
\[ K_S = -K(0) = \frac{K_{N1P} - \Gamma K_N}{1 - \Gamma} \quad \text{and} \quad K_G = K(\ell) = \frac{K_{N1P} + \Gamma K_N}{1 + \Gamma} . \quad (A-15) \]

The multifocus condition

The condition that an elementary wave traveling in the vicinity of the central ray focuses at a point \( P \) along the ray (see Figure 1) can be very simply translated into mathematically terms as the multifocus condition Gelchinsky et al. (1997)
\[ \frac{d\sigma_G}{d\sigma_S} = \frac{Q_G}{Q_S} . \quad (A-16) \]

Here, \( d\sigma_S \) and \( d\sigma_G \) are the arc elements of the wavefront at the source and receiver, respectively. The above condition follows from the definition of the dynamical quantity \( Q(s) \) as the square of the relative geometrical spreading computed at the point of the ray specified by \( s \) for a point source at the focus point \( P \). The consideration of the relative spreadings at the initial and end points of the central ray relative to the same point source at the focusing point \( P \), leads after a simple algebra to equation (A-16). We now observe that the above ratio between the \( Q \) variables at source and receiver can be readily computed as
\[ \frac{Q_G}{Q_S} \equiv \frac{Q(\ell)}{Q(0)} = aQ_N(\ell) + bQ_{N1P}(\ell) = \frac{aQ_N + bQ_{N1P}}{aQ_N - bQ_{N1P}} = \frac{\Gamma + 1}{\Gamma - 1} . \quad (A-17) \]
As a consequence, the multifocus condition assumes the form
\[ \frac{d\sigma_G}{d\sigma_S} = \frac{\Gamma + 1}{\Gamma - 1}. \]  
(A-18)

From geometrical considerations (see Figure 2), we have the relationship
\[ \frac{dx_S}{d\sigma_S} = \frac{1}{\cos \beta_S}. \]  
(A-19)

Substituting this expression and the corresponding equation for \(dx_G/d\sigma_G\) into equation (A-18) we obtain
\[ \frac{dx_G}{dx_S} = \frac{\Gamma + 1}{\Gamma - 1} \cdot \frac{\cos \beta_S}{\cos \beta_G}. \]  
(A-20)

The above differential equation cannot be solved exactly. Therefore, we will approximate the solution by its Taylor series up to the second-order, i.e.,
\[ \Delta x_G = \left. \frac{dx_G}{dx_S} \right|_{x_0} \Delta x_S + \frac{1}{2} \left. \frac{d^2x_G}{dx_S^2} \right|_{x_0} \Delta x_S^2, \]  
(A-21)

where the second coefficient can be determined by the derivative of equation (A-20). Thus, we need to determine the derivatives of \(\cos \beta_S\) and \(\cos \beta_G\) with respect to \(x_S\).

Again from Figure 2 we see that
\[ \cos \beta_S = \frac{R_S}{L_S} \cos \beta_0, \]  
(A-22)

where \(L_S = SC_S\). Therefore, the derivative with respect to \(x_S\) is
\[ \frac{d}{dx_S} \left[ \cos \beta_S \right] = \frac{-\cos \beta_S L_S}{L_S} \cdot \frac{dL_S}{dx_S}. \]  
(A-23)

Using that
\[ \frac{dL_S}{dx_S} = \frac{dSS'}{dx_S} = -\sin \beta_S, \]  
(A-24)

we find
\[ \frac{d}{dx_S} \left[ \cos \beta_S \right] = \frac{\sin \beta_S \cos \beta_S}{L_S}. \]  
(A-25)

An analogous equation holds for \(d \cos \beta_G/dx_G\). Moreover,
\[ \frac{d}{dx_S} \left[ \cos \beta_G \right] = \frac{d}{dx_G} \left[ \cos \beta_G \right] \cdot \frac{dx_G}{dx_S} = \frac{\sin \beta_G \cos \beta_S}{L_G} \cdot \frac{\Gamma + 1}{\Gamma - 1}. \]  
(A-26)

where we have used equation (A-20). The above results readily lead to the relation
\[ \frac{d}{dx_S} \left[ \frac{\cos \beta_S}{\cos \beta_G} \right] = \frac{\sin \beta_S}{L_S} \cdot \frac{\cos \beta_S}{L_S} - \frac{\sin \beta_G}{L_G} \cdot \left( \frac{\cos \beta_S}{\cos \beta_G} \right)^2 \cdot \frac{\Gamma + 1}{\Gamma - 1}. \]  
(A-27)
Computing the above expression on the central ray, i.e., $\beta_S = \beta_G = \beta_0$ as well as $L_S = R_S = 1/K_S$ and $L_G = R_G = 1/K_G$, we obtain
\[
\frac{d}{dx_S} \frac{\cos \beta_S}{\cos \beta_G} = \frac{2 K_{N1P} \sin \beta_0}{1 - \Gamma}.
\] (A-28)

Hence, in second-order approximation, the solution (A-21) of equation (A-20) reads
\[
\Delta x_G = \frac{\Gamma + 1}{\Gamma - 1} \Delta x_S \left[ 1 + \frac{\Delta x_S}{\Delta x_S} \frac{K_{N1P} \sin \beta_0}{1 - \Gamma} \right].
\] (A-29)

This equation describes the relationship between the source and receiver locations of all rays that cut the central ray at the same point $P$. Since $K_{N1P}$ and $\beta_0$ are parameters of the chosen central ray, the relation between $\Delta x_S$ and $\Delta x_G$ for a given focus point $P$ is solely determined by the value of $\Gamma$.

Conversely, equation (A-29) can be used to determine value of $\Gamma$ for any given ray with source at $S$ and receiver at $G$. Solving equation (A-29) for $\Gamma$ we find
\[
\Gamma = \Gamma_0 + \epsilon \frac{\Delta x_S}{\Delta x_G - \Delta x_S},
\] (A-30)

where $\Gamma_0$ is given by equation (9) in the text and
\[
\epsilon = \sqrt{(1 + \alpha \Delta x_S / 2)^3 - 2 \alpha \Delta x_G - (1 + \alpha \Delta x_S / 2)}.
\] (A-31)

Here, $\alpha$ is Gelchinsky’s asymmetry parameter, i.e.,
\[
\alpha = K_{N1P} \sin \beta_0.
\] (A-32)

Equation (A-30) is, in fact valid, up to second-order only. Thus, for consistency we have to replace equation (A-31) by its second-order Taylor series. We obtain
\[
\epsilon = -\Delta x_G \left( 1 + \frac{\Delta x_G - \Delta x_S}{2} \right) = -\Lambda (1 + \Lambda) \frac{2 \Delta x_S}{\Delta x_G - \Delta x_S},
\] (A-33)

where we have introduced the modified asymmetry parameter
\[
\Lambda = \frac{\alpha \Delta x_G - \Delta x_S}{2}.
\] (A-34)

Substituting expression (A-33) into formula (A-30), our final result for the multifocus parameter $\Gamma$ is equation (8) in the text, namely
\[
\Gamma = \Gamma_0 + \frac{\Lambda}{2} (1 + \Lambda) (1 - \Gamma_0^2).
\] (A-35)