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Weak–Contrast Edge and Vertex Diffractions in Anisotropic Elastic Media

Martin Tygel, and Bjørn Ursin[1]

ABSTRACT

Diffraction phenomena caused by edges and vertices are important in a number of studies and applications of wave propagation methods. Diffractions play a fundamental role, for example, in the interpretation of seismic data from faults and other complex geological structures in which petroleum reservoirs are located. The geometric theory of diffraction provides a good description of the diffraction of scalar waves from an edge in a homogeneous medium assuming free or rigid boundary conditions. The solution becomes infinite at the shadow boundary between the illuminated zone and the shadow zone, so that, within this region, a boundary layer solution must be used. This can be achieved by analytic continuation of the geometrical ray solution or, equivalently, using a one-dimensional or two-dimensional diffusion equation for modelling the amplitude of the diffracted wave. An alternative, older method for obtaining asymptotic expressions for different parts of the wavefield is by asymptotic expansion of the Kirchhoff integral. The authors have developed a new reciprocal surface-scattering integral by applying the divergence theorem to the Born volume integral. This new integral is called the Born-Kirchhoff (BK) integral. The scattering surface is a finite sum of smooth surfaces separated by smooth curves with a finite number of corners. The BK integral is now a sum of integrals over each smooth surface. These integrals are evaluated by the method of stationary phase, resulting in specularly reflected waves and boundary diffracted waves represented by a line integral along the boundary of each surface. Further asymptotic evaluation of these line integrals results in expressions for edge- and corner- diffracted waves. Smooth approximations of the wavefield are given for the case that a reflection point and an edge-diffraction point are close to each other, and when an edge-diffraction point and a corner-diffraction point are close to each other. These formulas can be cascaded to provide asymptotic expressions for multiple converted, reflected, transmitted and diffracted waves in anisotropic, inhomogeneous, elastic media. In the case that the rays are well separated from all shadow zones, the new expressions satisfy reciprocity.

[1] **email:** tygel@ime.unicamp.br

INTRODUCTION

Diffraction phenomena caused by edges and vertices are important in a number of studies and applications of wave propagation methods. Diffractions play a fundamental role, for example, in the interpretation of seismic data from faults and other complex geological structures in which petroleum reservoirs are located. Kanasewich and Phadke (1988) discuss this problem and give references to earlier work.

The case of scalar wave edge diffractions under free or rigid boundary conditions in homogeneous media is well described by the geometrical theory of diffractions (GTD) (Keller, 1962). This theory is based on Keller's law of diffraction, which states that an incident ray at a sharp edge generates a cone of diffracted rays. The amplitude of the diffracted wave depends on a diffraction coefficient, which can only be computed in special cases. These diffraction coefficients become infinitely large for diffracted rays in the vicinity of a shadow boundary between the illuminated zone and a shadow of the geometrical wavefield. This problem is overcome by using a shadow boundary layer solution (Buchal and Keller, 1960) for the diffracted and reflected wavefields near the shadow boundary. Achenbach and Gautesen (1977) have extended the GTD to the diffraction of three-dimensional longitudinal waves by a crack in an isotropic elastic medium.

Klem-Musatov and Aizenberg (1984) developed boundary-layer approximations for edge and vertex diffractions in isotropic elastic media. By analytic continuation, diffracted waves are derived which correct for the discontinuities in the geometric wavefield at the geometric shadow boundaries. The amplitude of the edge-diffracted wave can also be modelled by a one-dimensional transverse diffusion equation. The method has been used in the simulation of seismic waves in complex, three-dimensional geological models by Klem-Musatov, Aizenberg, Helle, and Pajchel (1995). Bakker (1990) has extended this theory by modelling the edge waves in the boundary zone, as a solution of a two-dimensional parabolic differential equation of transverse diffusion.

Van Kampen (1949) has given an asymptotic evaluation of the Kirchhoff integral for the reflected wavefield using the method of stationary phase. The Kirchhoff surface integral can be evaluated as a sum of contributions from stationary points in the interior of the surface, which correspond to the specularly reflected rays, as predicted by the geometric ray approximation (GRA) and an integral along the surface boundary, which corresponds to a boundary diffracted wave (BDW) (Yarygin, 1970; Stamnes, 1986; Hanyga, 1995). Van Kampen also showed that the surface boundary integral describing the BDW can be evaluated asymptotically as a superposition of edge-diffracted waves, corresponding to stationary points with respect to the boundary (critical points of the second kind), and corner diffractions, corresponding to corners at the surface boundary (critical points of the third kind).

Ursin and Tygel (1997) have recently developed a new reciprocal surface scattering integral, by applying the divergence theorem to the Born volume scattering integral in the case that changes in the medium parameters are confined to a smooth surface. Asymptotic evaluation of the surface integral resulted in the GRA for the reflected wavefield with a weak-contrast approximation of the linearized reflection coefficient. Our previous derivation can be easily extended to the case that the surface consists of a finite number of smooth surfaces. Then the total surface integral consists of a sum of integrals along the individual smooth surfaces. Following van Kampen (1949), each of these surface integrals can be evaluated asymptotically, resulting in a sum of reflected waves and BDWs, corresponding to integrating along the boundary contour of each surface. Each contour integral can be further evaluated asymptotically, resulting in edge- and corner-diffracted waves. Summing all individual contributions, results in an asymptotic evaluation of the total backscattered field.

In the following, we shall use results in Stamnes (1986) to give expressions for reflections, edge diffractions and corner diffractions when the critical points are well separated. These results correspond to leading-order terms in the high-frequency asymptotic expansions of standard, Fourier-type integrals. Stamnes (1986) also provides higher-order terms of the expansions, which, in principle, may yield more exact approximations of the integrals.

When two or more critical points are close, the previous expressions cannot be used and more general procedures must be applied (Stamnes, 1986; Hanyga, 1995). We shall only give expressions for the case that a simple reflection point and an edge-diffraction point coalesce, and the case that a simple edge-diffraction point and a corner-diffraction point coalesce. In fact, for these complicated cases, a direct numerical computation of the integral may be more advantageous.

We finally remark that, as GRA Green's functions are used in all derivations, the obtained expressions are not expected to provide a good description of the wavefield in the case the receiver lies within a caustic region.

The results can be cascaded to provide asymptotic expressions for multiple converted, reflected, transmitted and diffracted waves in anisotropic elastic media. In the case that all critical points are well separated (the rays are far away from any shadow boundary zone), it can be seen that the new expressions satisfy reciprocity. This means that by interchanging the source and receiver points, keeping all reflection, transmission and diffraction points at the boundaries, and having the same wave-mode in each block, results in the transposed Green's function.

THE SURFACE SCATTERING INTEGRAL

We consider the scattered wavefield in a medium where its parameters can be split into two parts corresponding to a background or reference medium and a perturbed medium. The parameters of the reference medium are usually smoothly varying functions, while the perturbations are considered to act as point scatterers or reflecting or refracting interfaces. In the background medium we shall use the GRA to obtain the approximate Green's function. Using the results in Chapman and Coates (1994) and Cervený (1995), we have for a specific ray connecting a source point $\mathbf{x}^s$ to a scattering point $\mathbf{x}$, the GRA Green's function

$$G^0_{ij}(\mathbf{x},\omega;\mathbf{x}^s) = \frac{h^s_i(\mathbf{x})}{[\rho^0(\mathbf{x})v^s(\mathbf{x})]^{1/2}} \left\{a(\mathbf{x},\mathbf{x}^s)\, e^{i\omega T(\mathbf{x},\mathbf{x}^s)}\right\} \frac{h_j(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)v(\mathbf{x}^s)]^{1/2}} \,, \quad (1)$$

where $\mathbf{h}(\mathbf{x}^s)$ and $\mathbf{h}^s(\mathbf{x})$ are the unit polarization vectors, $\rho^0(\mathbf{x})$ and $\rho^0(\mathbf{x}^s)$ are the densities and $v(\mathbf{x}^s)$ and $v^s(\mathbf{x})$ the phase velocities at the source and at the scattering point, respectively. Moreover, $T(\mathbf{x},\mathbf{x}^s)$ is the traveltime and

$$a(\mathbf{x},\mathbf{x}^s) = \frac{e^{-i\frac{\pi}{2}\,\sigma(\mathbf{x},\mathbf{x}^s)\,\mathrm{sgn}(\omega)}}{4\pi|\det\mathbf{Q}_2(\mathbf{x},\mathbf{x}^s)|^{1/2}} \quad (2)$$

is a complex amplitude function taking into account possible caustics. In this expression, $|\det\mathbf{Q}_2(\mathbf{x},\mathbf{x}^s)|^{1/2}$ denotes the relative geometric spreading factor and $\sigma(\mathbf{x},\mathbf{x}^s)$ is the KMAH index for the ray that connects the source $\mathbf{x}^s$ to the scattering point $\mathbf{x}$. Finally, $\mathrm{sgn}(\cdot)$ is the usual signum function

$$\mathrm{sgn}(\omega) = \begin{cases} -1\,, & (\omega < 0)\,, \\ +1\,, & (\omega \geq 0)\,. \end{cases} \quad (3)$$

In our previous paper Ursin and Tygel (1997), we investigated the single-scattering or Born response of a weak-contrast, anisotropic elastic medium in which the medium discontinuities were confined to a smooth interface. The main result is that this response can be represented by a surface integral representation along the discontinuity interface. Because of its resemblance to the Kirchhoff integral, we have designated this alternative representation as the Born-Kirchhoff (BK) integral. Please note that there is a formal error in our previous paper as corrected in Ursin and Tygel (1997). In the representation of the Green's function for the scattered field, the polarization vector and phase velocities at the source and receiver depend also on the direction of the ray to the scattering point. Therefore, the summations in the scattering integrals have to be made for each component of the Green's tensor.

In its original formulation, the discontinuity surface was supposed to be smooth and unbounded. We readily verify, however, that the BK integral representation holds true when the medium discontinuity surface is piecewise smooth, i.e., it is made up a finite number of smooth surfaces. Furthermore, each of these surfaces can be bounded by a piecewise smooth curve, i.e., made up of a finite number of smooth curves separated by corners. By linearity, the resulting BK representation will be simply the sum of the BK integrals over the individual surfaces. For fixed source and receiver position vectors x^s and x^r, respectively , and for a given circular frequency ω, we can write the BK integral contribution from surface component Σ as

$$\Delta G(\mathbf{x}^r, \omega; \mathbf{x}^s) = i\omega \int_\Sigma \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r) v(\mathbf{x}^r)]^{1/2}} \, b(\mathbf{x}) \, e^{i\omega T(\mathbf{x})} \, \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s) v(\mathbf{x}^s)]^{1/2}} \, d\boldsymbol{\sigma} \ , \tag{4}$$

where the function $b(\mathbf{x})$ is

$$b(\mathbf{x}) = S(\mathbf{x}) \, \frac{\boldsymbol{\nabla} T(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})}{|\boldsymbol{\nabla} T(\mathbf{x})|^2} \, \frac{a(\mathbf{x}, \mathbf{x}^r) \, a(\mathbf{x}, \mathbf{x}^s)}{[v^r(\mathbf{x}) v^s(\mathbf{x})]^{1/2}} \ , \tag{5}$$

in which $S(\mathbf{x})$ is the scattering function at the discontinuity surface given by

$$S(\mathbf{x}) = h_i^r(\mathbf{x}) \left[\frac{\rho^+(\mathbf{x}) - \rho^-(\mathbf{x})}{\rho^0(\mathbf{x})} \delta_{ik} + \frac{c_{ijkl}^+(\mathbf{x}) - c_{ijkl}^-(\mathbf{x})}{\rho^0(\mathbf{x})} p_j^r p_l^s \right] h_k^s(\mathbf{x}) \ , \tag{6}$$

and c_{ijkl} is the elastic stiffness tensor and the indices $\pm$ refer to the medium parameters beneath and above the surface Σ. The vectors $\mathbf{p}^s$ and $\mathbf{p}^r$ are the slowness vectors at the scattering point for the rays coming from the source and the receiver, respectively. The total traveltime is

$$T(\mathbf{x}) = T(\mathbf{x}, \mathbf{x}^s) + T(\mathbf{x}, \mathbf{x}^r) \ . \tag{7}$$

In the scattering integral (4), the polarization vectors $\mathbf{h}(\mathbf{x}^s)$ and $\mathbf{h}(\mathbf{x}^r)$ at the source and receiver, respectively, depend on direction, so they also depend on the scattering point $\mathbf{x}$. The same is the case for the phase velocities $v(\mathbf{x}^s)$ and $v(\mathbf{x}^r)$. For the sake of simplicity, we have not written this dependency on $\mathbf{x}$ explicitly.

The geometry for surface scattering is shown in Figure 1, where θ^s is the angle between the ray from the source and the surface normal, θ^r is the corresponding angle for the ray from the receiver and θ^{rs} is the angle between the two rays at the surface. Note that in the general, three-dimensional case, $\theta^{rs} \neq \theta^r + \theta^s$, except for the specular reflection point, where Snell's law is satisfied. As shown in Ursin and Tygel (1997), we have that

$$\boldsymbol{\nabla} T(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \frac{\cos\theta^r}{v^r} + \frac{\cos\theta^s}{v^s} \ , \tag{8}$$

and

$$|\boldsymbol{\nabla} T(\mathbf{x})|^2 = \frac{1}{(v^r)^2} + \frac{1}{(v^s)^2} + \frac{2\cos\theta^{rs}}{v^r v^s} \ , \tag{9}$$

which gives

$$b(\mathbf{x}) = S(\mathbf{x}) \, \frac{v^s \cos\theta^r + v^r \cos\theta^s}{[v^r v^s]^{1/2} [v^s/v^r + v^r/v^s + 2\cos\theta^{rs}]} \, a(\mathbf{x}, \mathbf{x}^r) a(\mathbf{x}, \mathbf{x}^s) \ . \tag{10}$$

DECOMPOSITION OF THE SCATTERED FIELD

The scattered field is computed from the surface integral given in equation (4) Classical analysis of this integral Stamnes (1986) shows that the main contributions to its value come from the singularities of the phase function $T(\mathbf{x})$ given in equation (7). These may be classified into:

(i) Critical points of the first kind: These are points in the interior or on the boundary of the surface Σ for which

$$\frac{\partial T}{\partial \sigma_j} = \boldsymbol{\nabla} T \cdot \mathbf{t}_j = 0 \ , \tag{11}$$

where $\mathbf{t}_j$, $j = 1, 2$, denote independent unit surface vectors from the surface parametrization. This condition is Snell's law, so that the critical points of the first kind provide, asymptotically, the reflections as described by the GRA Cervený (1995); Stamnes (1986).

(ii) Critical points of the second kind: These are points on the boundary Γ of the surface Σ at which

$$\frac{\partial T}{\partial \gamma} = \boldsymbol{\nabla} T \cdot \mathbf{t} = 0 \ , \tag{12}$$

where $\mathbf{t}$ is the boundary tangent unit vector. In other words, the directional derivative of the phase along the boundary vanishes. This condition can be recognized as the well-known extension of Snell's law, which is appropriate for edge diffractions Keller (1962). Critical points of the second kind give rise asymptotically to the contributions to the total field coming from smooth edges, as predicted by the GTD.

(iii) Critical points of the third kind: These are points on the boundary Γ of the surface Σ where

$$\left.\frac{\partial T}{\partial \gamma}\right|_{-} \neq \left.\frac{\partial T}{\partial \gamma}\right|_{+} \ , \tag{13}$$

i.e., the derivative along the boundary suffers a jump discontinuity. The typical situation is that of a corner. Critical points of the third kind are responsible, asymptotically, for the contributions to the total field corresponding to corner or vertex diffractions.

In the following, we shall first give the contributions to the scattered field when the singular points are well separated. When two or more critical points are close together, this simple analysis is no longer valid, and the combined contribution to the scattered field from these critical points must be evaluated.

SPECULAR REFLECTIONS

Critical points of the first kind give rise to specular reflections corresponding to the GRA. Assuming that the critical points are well separated, we may write the BK surface integral as a sum of contributions from the interior stationary points, corresponding to specular reflections, and a line integral along Γ, the boundary of Σ. For each stationary interior point $\mathbf{x}^R$ where the condition (11) is fulfilled, and which is regular so that $\det \mathbf{H}(\mathbf{x}^R) \neq 0$, where the matrix $\mathbf{H}$ has elements

$$H_{ij} = \frac{\partial^2 T}{\partial \sigma_i \partial \sigma_j} = \frac{\partial x_n}{\partial \sigma_i} \frac{\partial x_k}{\partial \sigma_j} + \frac{\partial T}{\partial x_k} \frac{\partial^2 x_k}{\partial \sigma_i \partial \sigma_j} , \quad i, j = 1, 2 , \tag{14}$$

the asymptotic evaluation of the integral (4) leads to the expression (Ursin and Tygel, 1997, equations (48)-(52))

$$\begin{aligned} \Delta G_R(\mathbf{x}^r, \omega, \mathbf{x}^s) &= \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r) v(\mathbf{x}^r)]^{1/2}} \left\{ \frac{R(\mathbf{x}^R)\ e^{-i\frac{\pi}{2}\sigma_R(\mathbf{x}^R)\,\mathrm{sgn}(\omega)}}{4\pi \mathcal{L}_R(\mathbf{x}^R)}\ e^{i\omega T(\mathbf{x}^R)} \right\} \\ &\times \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s) v(\mathbf{x}^s)]^{1/2}} . \end{aligned} \tag{15}$$

This is the GRA Green's function for the reflected wave with reflection coefficient

$$R(\mathbf{x}^R) = -\ S(\mathbf{x}^R) \frac{[\tan \theta^r \tan \theta^s]^{1/2}}{2 \sin(\theta^r + \theta^s)} , \tag{16}$$

relative geometrical spreading factor

$$\mathcal{L}_R(\mathbf{x}^R) = |\det \mathbf{Q}_2(\mathbf{x}^R)|^{1/2} = \left| \frac{\det \mathbf{H}(\mathbf{x}^R)\ \det \mathbf{Q}_2(\mathbf{x}^R, \mathbf{x}^r)\ \det \mathbf{Q}_2(\mathbf{x}^R, \mathbf{x}^s)}{\cos \theta^r \cos \theta^s} \right|^{1/2} , \tag{17}$$

and KMAH index

$$\sigma_R(\mathbf{x}^R) = \sigma(\mathbf{x}^R, \mathbf{x}^r) + \sigma(\mathbf{x}^R, \mathbf{x}^s) + [1 - \mathrm{Sgn}(\mathbf{H}(\mathbf{x}^R))/2] . \tag{18}$$

In the above formulas, $\mathrm{Sgn}\,\mathbf{H}(\mathbf{x}^R)$ is the signature of the matrix $\mathbf{H}(\mathbf{x}^R)$, i.e., the difference between the number of its positive and the number of its negative eigenvalues. The negative sign in the expression for the reflection coefficient is due to the fact that the scattering coefficient used in the surface integral relates to two down-going waves (relative to the surface) while the reflection coefficient relates the down-going and up-going wave. This means that in equation (4), the polarization vector $\mathbf{h}^r(\mathbf{x})$, related to the ray from the receiver to the scattering point, must change sign for the reflected field in equation (15).

EDGE DIFFRACTIONS

In the asymptotic evaluation of equation (4), there will also be a contribution from the surface boundary Γ. This is given by the line integral Yarygin (1970); Stamnes (1986)

$$\begin{aligned} \Delta G_{BDW}(\mathbf{x}^r, \omega, \mathbf{x}^s) &= \int_\Gamma \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r)\, v(\mathbf{x}^r)]^{1/2}}\, b(\mathbf{x})\, \frac{\boldsymbol{\nabla} T(\mathbf{x}) \cdot \mathbf{m}(\mathbf{x})}{|\boldsymbol{\nabla} T(\mathbf{x})|^2}\, \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)\, v(\mathbf{x}^s)]^{1/2}} \\ &\times\ e^{i\omega T(\mathbf{x})}\, d\gamma, \end{aligned} \tag{19}$$

where the vector $\mathbf{m}$ denotes the unit normal to the boundary Γ which is tangent to the surface Σ. This normal is supposed to exist for all points of Γ, except for a finite number of boundary corners. This boundary integral accounts for the BDW part of the total wavefield, and it can also be asymptotically evaluated. As a result, the individual contributions of the critical points of second and third kind can be singled out and interpreted as edge and corner diffractions.

At the edge, we shall define a local coordinate system $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, such that γ_1 points in the direction of the edge tangent $\mathbf{t}$, γ_2 points in the direction of vector $\mathbf{m}$, which is normal to the edge and in the surface tangent plane, and γ_3 points in the direction of the surface normal $\mathbf{n}$. The coordinate system $\boldsymbol{\gamma}$ is depicted in Figure 2. We shall also use the notation $T_{i,j} = \dfrac{\partial^{i+j} T}{\partial \gamma_1^i \, \partial \gamma_2^j}$.

At the critical points of the second kind $\mathbf{x}^E$, for which Keller's law of diffraction (compare with equation (12))

$$T_{1,0} = \frac{\partial T}{\partial \gamma_1} = 0 \tag{20}$$

is fulfilled, and which are regular so that

$$T_{2,0} = \frac{\partial^2 T}{\partial \gamma_1^2} = \frac{\partial^2 T}{\partial x_i \partial x_j} \, \frac{\partial x_i}{\partial \gamma_1} \, \frac{\partial x_j}{\partial \gamma_1} + \frac{\partial T}{\partial x_i} \, \frac{\partial^2 x_i}{\partial \gamma_1^2} \tag{21}$$

is different from zero, we obtain an asymptotic contribution (Stamnes, 1986, equations (9.47)) which corresponds to the edge-diffracted wave (see equations (4) and (5))

$$\begin{aligned} \Delta G_E(\mathbf{x}^r, \omega, \mathbf{x}^s) \;\; = \;\; & \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r) v(\mathbf{x}^r)]^{1/2}} \left\{ \frac{[\boldsymbol{\nabla} T(\mathbf{x}^E) \cdot \mathbf{n}(\mathbf{x}^E)][\boldsymbol{\nabla} T(\mathbf{x}^E) \cdot \mathbf{m}(\mathbf{x}^E)]}{|T_{2,0}(\mathbf{x}^E)|^{1/2} |\boldsymbol{\nabla} T(\mathbf{x}^E)|^4 [v^r(\mathbf{x}^E) v^s(\mathbf{x}^E)]^{1/2}} \; S(\mathbf{x}^E) \right. \\ & \times \; \left. \left| \frac{2\pi}{\omega} \right|^{1/2} e^{i \frac{\pi}{4} \mathrm{sgn}(\omega \, T_{2,0}(\mathbf{x}^E))} a(\mathbf{x}^E, \mathbf{x}^r) a(\mathbf{x}^E, \mathbf{x}^s) e^{i\omega T(\mathbf{x}^E)} \right\} \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s) v(\mathbf{x}^s)]^{1/2}} \; , \end{aligned} \tag{22}$$

This result may be interpreted as

$$\Delta G_E(\mathbf{x}^r, \omega, \mathbf{x}^s) = \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r) v(\mathbf{x}^r)]^{1/2}} \left\{ \frac{E(\mathbf{x}^E) \; e^{-i \, \frac{\pi}{2} \sigma_E(\mathbf{x}^E) \mathrm{sgn}(\omega)}}{4\pi \mathcal{L}_E(\mathbf{x}^E)} \; e^{i\omega T(\mathbf{x}^E)} \right\} \; \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s) v(\mathbf{x}^s)]^{1/2}} \; , \tag{23}$$

where the edge-diffraction coefficient is

$$E(\mathbf{x}^E) = - \; \frac{1}{\sqrt{8\pi |\omega|}} \; \frac{[\boldsymbol{\nabla} T(\mathbf{x}^E) \cdot \mathbf{n}(\mathbf{x}^E)][\boldsymbol{\nabla} T(\mathbf{x}^E) \cdot \mathbf{m}(\mathbf{x}^E)]}{|\boldsymbol{\nabla} T(\mathbf{x}^E)|^4} \; \frac{S(\mathbf{x}^E)}{|v(\mathbf{x}^r) \, v(\mathbf{x}^s) \sin \beta^r \sin \beta^s|^{1/2}} \; , \tag{24}$$

the relative geometrical spreading is

$$\mathcal{L}_E(\mathbf{x}^E) = \left| \frac{\det Q_2(\mathbf{x}^E, \mathbf{x}^r) \; \det Q_2(\mathbf{x}^E, \mathbf{x}^s) \; T_{2,0}(\mathbf{x}^E)}{\sin \beta^r \; \sin \beta^s} \right|^{1/2} \; , \tag{25}$$

and the corresponding KMAH-index is

$$\sigma_E(\mathbf{x}^E) = \sigma(\mathbf{x}^E, \mathbf{x}^r) + \sigma(\mathbf{x}^E, \mathbf{x}^s) + [2 - \mathrm{sgn}(T_{2,0}(\mathbf{x}^E))/2] \ . \tag{26}$$

In the above expressions, β^r and β^s are the angles between the tangent of the curve Γ with the rays from the receiver point $\mathbf{x}^r$ and the source point $\mathbf{x}^s$, respectively. Note that this expression is very similar to the expression in equation (17) for the relative spreading for the specular reflected wave. In the plane of reflection, we have that $|\sin\beta^r \sin\beta^s| = |\cos\theta^r \cos\theta^s|$.

CORNER DIFFRACTIONS

For each corner $\mathbf{x}^C$ of the surface boundary Γ, which is regular so that $T_{1,0}(\mathbf{x}^C) \neq 0$, we obtain two asymptotic contributions to the line integral (19) (Stamnes, 1986, equations (8.14)). These give rise to the corner diffracted wave (see equations (4) and (5))

$$\Delta G_C(\mathbf{x}^r, \omega, \mathbf{x}^s) = \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r)v(\mathbf{x}^r)]^{1/2}} \left\{ \frac{S(\mathbf{x}^C)}{i\omega} \frac{[\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot\mathbf{n}(\mathbf{x}^C)][\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot\mathbf{m}(\mathbf{x}^C)]}{T_{1,0}(\mathbf{x}^C)|\boldsymbol{\nabla} T(\mathbf{x}^C)|^4\, [v^r(\mathbf{x}^C)v^s(\mathbf{x}^C)]^{1/2}} \right. \tag{27}$$

$$\left. \times \quad a(\mathbf{x}^C, \mathbf{x}^r)\ a(\mathbf{x}^C, \mathbf{x}^s)\ e^{i\omega T(\mathbf{x}^C)} \right\} \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)v(\mathbf{x}^s)]^{1/2}} \ ,$$

In the formulas above, the derivative $T'(\mathbf{x}^C)$ is to be computed in the direction of the corner. At a corner, there will be two such contributions coming from two different line integrals. The sum of them is given in (Stamnes, 1986, equation (9.43)). As before, this result above may be interpreted as

$$\Delta G_C(\mathbf{x}^r, \omega, \mathbf{x}^s) = \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r)v(\mathbf{x}^r)]^{1/2}} \left\{ \frac{C(\mathbf{x}^C)\ e^{-i\,\frac{\pi}{2}\sigma_C(\mathbf{x}^C)\,\mathrm{sgn}(\omega)}}{4\pi\mathcal{L}_C(\mathbf{x}^C)}\ e^{i\omega T(\mathbf{x}^C)} \right\} \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)v(\mathbf{x}^s)]^{1/2}} \ , \tag{28}$$

where the corner-diffraction coefficient is

$$C(\mathbf{x}^C) = -\,\frac{1}{4\pi|\omega|}\ \frac{[\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot\mathbf{n}(\mathbf{x}^C)][\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot\mathbf{m}(\mathbf{x}^C)]}{|\boldsymbol{\nabla} T(\mathbf{x}^C)|^4}\ \frac{S(\mathbf{x}^C)}{|v(\mathbf{x}^r)\ v(\mathbf{x}^s)\sin\beta|^{1/2}} \ , \tag{29}$$

the relative geometrical spreading is

$$\mathcal{L}_C(\mathbf{x}^C) = \left| \frac{\det Q_2(\mathbf{x}^C, \mathbf{x}^r)\ \det Q_2(\mathbf{x}^C, \mathbf{x}^s)\ T^2_{1,0}(\mathbf{x}^C)}{\sin\beta} \right|^{1/2} \ , \tag{30}$$

and the corresponding KMAH-index is

$$\sigma_C(\mathbf{x}^C) = \sigma(\mathbf{x}^C, \mathbf{x}^r) + \sigma(\mathbf{x}^C, \mathbf{x}^s) - \mathrm{sgn}(T_{1,0}(\mathbf{x}^C)) \ . \tag{31}$$

In a practical situation when we are modelling an elastic solid, there may be several surfaces sharing the same boundary Γ, and then the total edge diffraction will be a sum of terms from the different surface integrals. In the same way, the diffraction from a vertex will be a sum of corner diffractions from several boundaries.

SIMPLE REFLECTION BOUNDARY ZONE

The previous asymptotic expressions were derived under the assumption that all critical points are sufficiently separated from one another. Now we shall consider the case when an interior stationary point is near a stationary point of the second kind on the boundary. In other words, a reflection point x^R is near an edge-diffraction point x^E. In that case, we can approximate the surface integral in equation (4) using equations (9.58) in Stamnes (1986). This gives, in our notation,

$$
\begin{aligned}
\Delta G_{RE}(\mathbf{x}^r,\omega,\mathbf{x}^s) &= \Delta G_R(\mathbf{x}^r,\omega,\mathbf{x}^s)\, H(-\gamma_2) \\
&+ \operatorname{sgn}(\gamma_2)\frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r)v(\mathbf{x}^r)]^{1/2}} \Big\{ a(\mathbf{x}^E,\mathbf{x}^r)a(\mathbf{x}^E,\mathbf{x}^s) \\
&\times \frac{\boldsymbol{\nabla} T(\mathbf{x}^E)\cdot \mathbf{n}(\mathbf{x}^E)}{|\boldsymbol{\nabla}(\mathbf{x}^E)|^2}\, \frac{S(\mathbf{x}^E)\ \pi}{[v^r(\mathbf{x}^E)\, v^s(\mathbf{x}^E)]^{1/2}}\, \frac{e^{i\,\frac{\pi}{4}[\operatorname{Sgn}(\mathbf{H}(\mathbf{x}^E))+2]\operatorname{sgn}(\omega)}}{|\det \mathbf{H}(\mathbf{x}^E)|^{1/2}} \\
&\times e^{i\omega T(\mathbf{x}^E)}\ e^{-i\omega\nu_E}\operatorname{erfc}\left[|\omega\,\nu_E|^{1/2} e^{-i\,\frac{\pi}{4}\operatorname{sgn}(\omega\,\nu_E)}\right]\Big\}\, \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)v(\mathbf{x}^s)]^{1/2}},
\end{aligned} \tag{32}
$$

where

$$
\nu_E = \frac{[T_{0,1}(\mathbf{x}^E)]^2}{2\,T_{0,2}(\mathbf{x}^E)} \tag{33}
$$

and $H(x)$ is the Heaviside function

$$
H(x) = \begin{cases} 0\,, & (x < 0)\,, \\ 1\,, & (x \geq 0)\,. \end{cases} \tag{34}
$$

Finally, erfc(x) is the complementary error function given by

$$
\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s^2}\ ds\ . \tag{35}
$$

This function is closely related to the Fresnel integrals that often appear in diffraction problems (Stamnes, 1986, equations (8.24)). The parameter $\operatorname{sgn}(\gamma_2)$ is minus one in the illuminated zone where there is a reflection, and it is equal to one in the shadow zone of the reflection. In order to compare equation (9.58) in Stamnes (1986) with equation (32) above, it is important to note that the variables F_{ij} in Stamnes are mixed-derivatives divided by $(i+j)!$ to simplify his expressions. Furthermore, the variables X and Y in Stamnes correspond to our variables $-\gamma_2$ and γ_1, respectively.

As shown in (Stamnes, 1986, section 9.2.2), this scattered wavefield has the proper asymptotic behaviour. As $\mathbf{x}^R \rightarrow \mathbf{x}^E$, we obtain the wavefield on the shadow boundary, namely

$$
\Delta G_{RE} \rightarrow \frac{1}{2}\,\Delta G_R\ . \tag{36}
$$

As the stationary point $\mathbf{x}^R$ moves away from the boundary, the complementary error function can be replaced by its first term in its asymptotic expansion. This gives

$$
\Delta G_{RE} \approx \Delta G_R\ H(\varepsilon_R) + \Delta G_E\ , \tag{37}
$$

where ΔG_E is the contribution from the edge-diffracted wave given in equation (23). The parameter ε_R equals one in the zone where there is a reflected wave, and equals minus one in the shadow zone.

SIMPLE DIFFRACTION BOUNDARY ZONE

When an edge diffraction point $\mathbf{x}^E$ is close to a corner $\mathbf{x}^C$, the previous expressions for the edge diffraction and corner diffraction can no longer be used. In that case, the contribution to the line integral in equation (19) can be approximated using equation (8.28) in Stamnes (1986). This provides, in our notation,

$$\begin{aligned} \Delta G_{EC}(\mathbf{x}^r,\omega,\mathbf{x}^s) &= \Delta G_E(\mathbf{x}^r,\omega,\mathbf{x}^s)\, H(-\gamma_1) \\ &+ \operatorname{sgn}(\gamma_1)\, \frac{\mathbf{h}(\mathbf{x}^r)}{[\rho^0(\mathbf{x}^r)v(\mathbf{x}^r)]^{1/2}} \left\{ a(\mathbf{x}^C,\mathbf{x}^r)a(\mathbf{x}^C,\mathbf{x}^s) \left| \frac{\pi}{2\omega\, T_{2,0}(\mathbf{x}^C)} \right|^{1/2} \right. \\ &\times \frac{[\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot \mathbf{n}(\mathbf{x}^C)][\boldsymbol{\nabla} T(\mathbf{x}^C)\cdot \mathbf{m}(\mathbf{x}^C)]}{|\boldsymbol{\nabla} T(\mathbf{x}^C)|^4}\, \frac{S(\mathbf{x}^C)}{[v^r(\mathbf{x}^C)\, v^s(\mathbf{x}^C)]^{1/2}}\, e^{i\frac{\pi}{4}\operatorname{sgn}(\nu_C\omega)} \\ &\times \left. e^{i\omega\, T(\mathbf{x}^C)}\, e^{-i\omega\nu_C}\, \operatorname{erfc}\left[|\omega\,\nu_C|^{1/2}\, e^{-i\,\frac{\pi}{4}\operatorname{sgn}(\omega\,\nu_C)} \right] \right\} \frac{\mathbf{h}^T(\mathbf{x}^s)}{[\rho^0(\mathbf{x}^s)v(\mathbf{x}^s)]^{1/2}} \,, \end{aligned} \tag{38}$$

where

$$\nu_C = \frac{[T_{1,0}(\mathbf{x}^C)]^2}{2\, T_{2,0}(\mathbf{x}^C)} \,. \tag{39}$$

The parameter $\operatorname{sgn}(\gamma_1)$ is minus one in the illuminated zone, where there is an edge diffraction, and it is one in the shadow zone of the edge diffraction.

It is shown in (Stamnes, 1986, equation (8.31)) that this diffracted wavefield has the correct asymptotic behaviour. As $\mathbf{x}^E \to \mathbf{x}^C$, we obtain the diffracted wavefield on the diffraction shadow boundary, as given by

$$\Delta G_{EC} \to \frac{1}{2}\, \Delta G_E \,. \tag{40}$$

Furthermore, as the edge-diffraction point moves away from the corner point, the complementary error function can be replaced by the first term in its asymptotic expansion. This gives

$$\Delta G_{EC} \approx \Delta G_E\, H(\varepsilon_E) + \Delta G_C \,, \tag{41}$$

where ΔG_C is the contribution to the corner diffraction given in equation (28). The parameter ε_E is now equal to one where there is an edge diffraction and equal to minus one in the diffraction shadow zone.

SIMPLE REFLECTION AND DIFFRACTION BOUNDARY ZONE

We have now considered all simple cases for the critical points, except the one of a simple reflection and diffraction boundary zone. This occurs when two edge-diffraction points and a reflection point coalesce near a corner point. When all these points are well separated, we can add our previous expressions. The scattered wavefield can then be written as

$$\Delta G_D = \Delta G_R + \Delta G_E^1 + \Delta G_C^1 + \Delta G_E^2 + \Delta G_C^2 \ , \tag{42}$$

where ΔG_R is the reflected wavefield in equation (15), ΔG_E^1 and ΔG_E^2 are the edge diffractions, as given by equation (23), from the two parts of the boundary which form the corner, and ΔG_C^1 and ΔG_C^2 are the two end-point contributions, as given by equation (28). When the critical points are close together, there is a single reflected-diffracted field, as given by Hanyga (1995).

CONCLUSIONS

Asymptotic evaluation of the BK surface integral resulted in the GRA for the reflected wave and a new line integral for the BDW. Further asymptotic evaluation of the BDW integral, resulted in new expressions for the edge-diffracted and corner-diffracted waves. Smooth approximations of the wavefield were given for the case that a reflection point and an edge-diffraction point are close to each other, and when an edge-diffraction point and a corner-diffraction point are close to each other.

These formulas can be cascaded to provide asymptotic expressions for multiple converted, reflected, transmitted and diffracted waves in anisotropic, inhomogeneous, elastic media. In the case that the rays are well separated from all shadow zones, the new expressions satisfy reciprocity.

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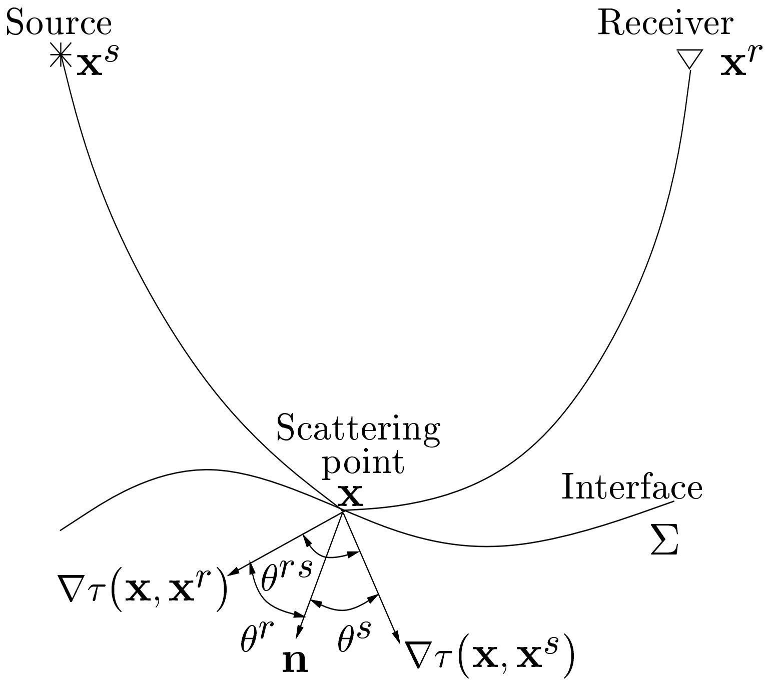

Figure 1: Geometry for surface scattering.

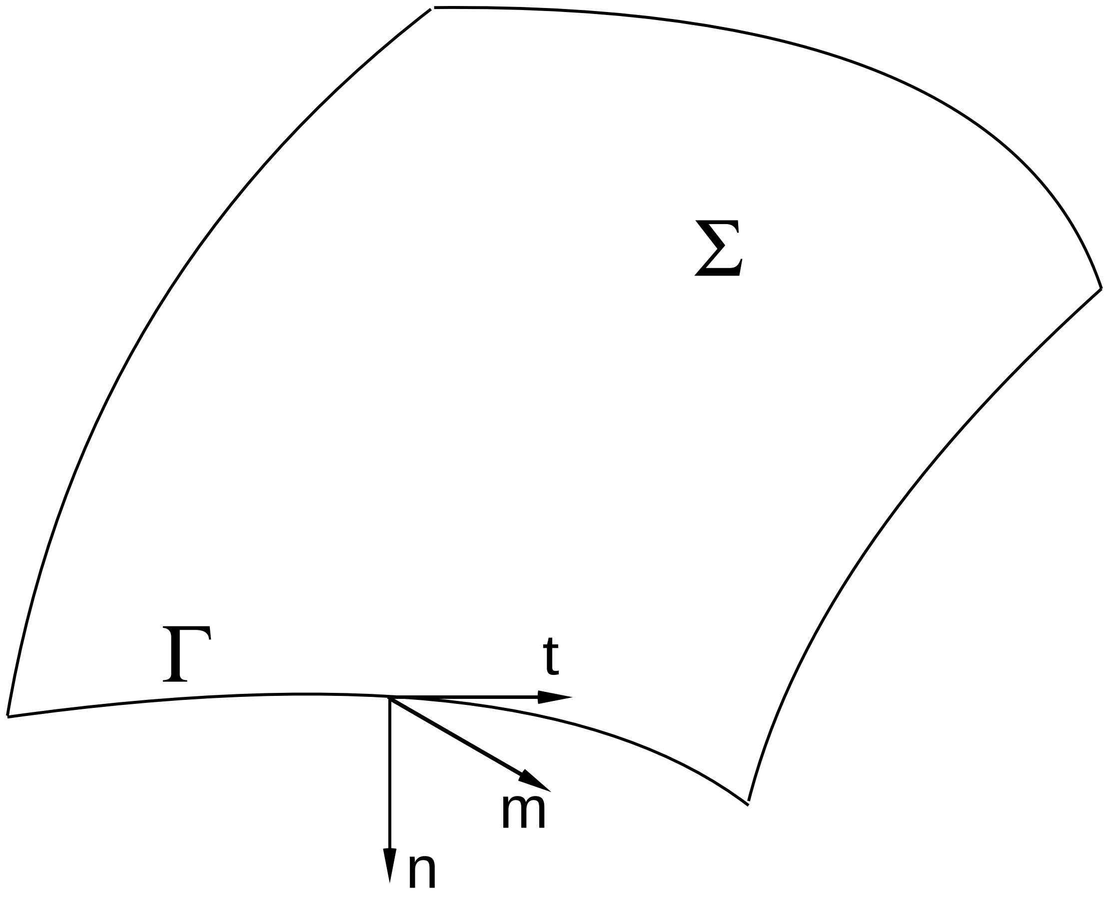

Figure 2: Geometry of the edge variables.