Wave Inversion Technology, Report No. 2, pages 67-87

The Kirchhoff-Helmholtz Integral Pair

M. Tygel, L. T. Santos, J. Schleicher, and P. Hubral[1]

keywords: *Kirchhoff approximation, inverse Kirchhoff integral*

ABSTRACT

Modeling a reflected wave by the Kirchhoff-Helmholtz integral consists of an integration along the reflector. By this, one sums up the Huygens secondary-source contributions to the wavefield attatched to the reflector at the observation point. The proposed asymptotic inverse Kirchhoff-Helmholtz integral, by which this well-known modeling process is inverted, works in a completely analogous way. It consists of an integral along the reflection traveltime surface of the reflector. For a point on the reflector, one sums up (integrates) the reflected-wave contributions attached to the respective reflection-traveltime surface associated with the related source-receiver pair. In this way, the new integral is a more natural inverse to Kirchhoff-Helmholtz forward modeling integral than the conventional Kirchhoff migration integral. Like the latter, the new inverse integral reconstructs the Huygens sources along the reflector, thus providing their positions and amplitudes. This enables the realization of important aspects of wave inversion closely related to, but nevertheless quite different from, the conventional Kirchhoff migration process well-known in seismic reflection imaging.

INTRODUCTION

The classical Kirchhoff integral solution of the acoustic wave equation describes the response of a given wavefield (e.g. originating from a well-specified source), measured upon a closed surface, at a given observation point within the volume enclosed by that surface (see, e.g., Bleistein, 1984). In different approximations, the Kirchhoff integral provides a well-known useful tool to numerically simulate the wave propagation. For instance, the wavefield originating from a point source and primarily reflected from a smooth reflector overlain by a smooth inhomogeneous acoustic medium can be described by the Kirchhoff integral in the so-called single-scattering, high-frequency approximation (see, e.g., Bleistein, 1984; Frazer and Sen, 1985). The resulting Kirchhoff-Helmholtz integral describes then the reflected elementary waves as a superposition of Huygens secondary point sources distributed along the reflector. A

[1] **email:** tygel@ime.unicamp.br

useful picture of this process is to imagine that the reflector is made up by point-source diffractors that are idividualy excited by the incoming of the incident wavefield. The intensity of each of these point diffractors along the reflector, regarded a weight in the integral, is specified by the so-called Kirchhoff-Helmholtz approximation. It are given by the product of the incident field and the plane-wave reflection coefficient relative to the incident ray that connects the source to the diffraction point, all these quantities being computed at this point. The weavefields radiated by all Huygens, point-diffraction sources contribute in form of a constructive interference to the wave measured at the receiver. The asymptotic evaluation of the Kirchhoff-Helmholtz integral leads to the familiar zero-order ray theory approximation (or geometrical optics solution) of the reflected wavefield at the receiver (Tygel et al., 1994, see also references there). Upon proper modification of the weight function along the Kirchhoff-Helmholtz integral, a similar representation integral can be constructed to account for the propagation of diffraction waves at edges and corners at the reflector (see, e.g, Klem-Musatov and Aizenberg, 1984; Tygel and Ursin, 1998). By variation of the locations of the sources and receivers according to a chosen measurement configuration, a seismic multicoverage experiment can be simulated. The primary reflections due to the given reflector, align along the corresponding, so-called reflection-traveltime surface. This surface can be interpreted as the kinematic image of the reflector in the seismic record section associated with the chosen configuration. In other words, we can say that this surface implicitly results from the evaluation of the Kirchhoff-Helmholtz integral. In the same way, the wave observed along this surface dynamically images the Huygens sources.

The Kirchhoff-Helmholtz integral is largely used to accurately model primary reflections in smoth layered models bounded by smooth interfaces (reflectors). A natural question that arises is whether a transformation exists that performs the opposite task of the Kirchhoff-Helmholtz integral. In other words, this inverse would have to *kinematically and dynamically reconstruct* the reflector. This would have to involve a weighted superposition of the observed elementary wave along the reflection traveltime surface of the searched-for reflector. To kinematically and dynamically reconstruct the reflector means to asymptotically recover the reflector location together with the plane-wave reflection coefficient and each point of the reflector. In the seismic literature, this is commonly called the *true amplitude* at all reflector points.

The problem of reonstructing a subsurface reflector out of seismic reflection records on a given configuration is called in the seismic literature the *depth-migration problem* (see, e.g. Stolt and Benson, 1986). The depth migration method that is traditionally accepted as an inverse to the Kirchhoff-Helmholtz integral is the Kirchhoff depth migration (Schneider, 1978, Stolt and Benson, 1986). The Kirchhoff depth migration is realized upon summing up contributions of the reflection data along auxiliary diffraction surfaces constructed on an a priori, given reference model. The basic idea of the Kirchhoff depth migration is that points on the reflector give rise to consructed diffraction traveltime surfaces that are tangent to the corresponding observed primary reflection traveltime surfaces. Because of phase coherence along the tangential region, the summation process accumulates constructively interfering signals to provide a significantly larger amplitude than the counterpart amplitudes obtained when the summation process is carried out for point-diffraction surfaces away from the reflectors. Each difference diffraction traveltime surface refers to a specific point in depth, on which the resulting summed amplitude is assigned. Performing the above procedure to a fine grid on depth provides the subsurface depth-migration image.

From the above descriptions of the Kirchhoff-Helmholtz integral and the Kirchhoff depth migration, we readily realize that both methods are structurely very different. The former is a *modeling process* that uses the actual reflector as a superposition integral. The latter is an *imaging process* that superposes the data along prescribed surfaces constructed on a reference model. As shown in Hubral et al. (1996) and Tygel et al. (1996), the Kirchhoff depth migration is the (asymptotic) inverse of a different operation called *demigration*, realized upon summation on depth-domain surfaces (isochrones) constructed on the same reference model.

So far, the Kirchhoff-Helmholtz integral, a summation operator along a given reflector, lacks a structurely similar (asymptotic) inverse operation. This should have the form of a summation operation along the reflection traveltime corresponding the reflector, assuming, of course, the same configuration of source- and -receivers pairs.

This is being set up in this paper by exploring the dual properties between the given reflector and its corresponding traveltime surface. The resulting new inverse Kirchhoff-Helmholtz integral is then completely analogous to the well-known, forward-modeling, Kirchhoff-Helmholtz integral.

FORMULATION OF THE PROBLEM

To formulate the Kirchhoff-Helmholtz integral transformation pair, we make the following assumptions about the model and the 3-D wave-propagation:

- We assume the model of a smoothly varying inhomogeneous acoustic medium, bounded above and below by two smooth surfaces. The upper one is the measurement surface Σ_M and the lower one is the target reflector Σ. The target reflector Σ is parameterized as $z = \Sigma(\mathbf{x})$, in which $\mathbf{x}$ is the two-dimensional horizontal coordinate vector varying on the spatial aperture set E. Points on the reflector Σ will be generally denoted by $M = M(\mathbf{x}) = (\mathbf{x}, z = \Sigma(\mathbf{x}))$, parameterized by $\mathbf{x}$ in E.

- A dense distribution of sources and receivers is specified in a certain area of the measurement surface Σ_M parametrized as $z = \Sigma_M(\mathbf{x})$. The sources and receivers are grouped in pairs as described by the measurement configuration involved. The locations of the source-receiver pairs are given as a function of a two-dimensional vector parameter $\boldsymbol{\xi}$ that varies on a given configuration aperture set A. In other words, each source at point $S = S(\boldsymbol{\xi}) = (\mathrm{x}_S(\boldsymbol{\xi}), z_S(\boldsymbol{\xi}) = \Sigma_M(\mathrm{x}_S(\boldsymbol{\xi})))$ corresponds to exactly one receiver at position $G = G(\boldsymbol{\xi}) = (\mathrm{x}_G(\boldsymbol{\xi}), z_G(\boldsymbol{\xi}) = \Sigma_M(\mathrm{x}_G(\boldsymbol{\xi})))$, with $\boldsymbol{\xi}$ in A.

- For each source-receiver pair, there exists one and only one point $M_R = M(\boldsymbol{x}_R) = (\boldsymbol{x}_R, z = \Sigma(\boldsymbol{x}_R))$ on the reflector Σ, for which the composite ray SM_RG describes a specular primary reflection. The dependency $\boldsymbol{x}_R = \boldsymbol{x}_R(\boldsymbol{\xi})$ implies that the location of the specular reflection point M_R is determined by the location of the source-receiver pair (S, G), which in turn is specified by $\boldsymbol{\xi}$. We will denote the plane-wave reflection coefficient for the ray SM_RG at M_R by $\mathcal{R}(M_R)$.

- The function $t = \Gamma(\boldsymbol{\xi})$, for varying $\boldsymbol{\xi}$ in A, describes the reflection traveltime from the source $S(\boldsymbol{\xi})$ to the receiver $G(\boldsymbol{\xi})$ along along the primary-reflection ray SM_RG. This function is called the reflection-traveltime surface Γ of the target reflector Σ. Both surfaces are said to be duals of each other. Points on the traveltime surface Γ will be denoted by $N = N(\boldsymbol{\xi}) = (\boldsymbol{\xi}, t = \Gamma(\boldsymbol{\xi}))$.

- For each point M on the target reflector Σ, there exists one and only one source-receiver pair (S_R, G_R) for which the composite ray S_RMG_R pertains to a specular primary reflection at M. This pair (S_R, G_R) is parameterized by a fixed value of $\boldsymbol{\xi}_R = \boldsymbol{\xi}_R(\mathrm{x})$ depending on the horizontal coordinate x of M. Note that this coordinate defines M as a point on the reflector Σ. We will denote the plane-wave reflection coefficient for the ray S_RMG_R at M by $\mathcal{R}(M)$. Also, the notation $N_R = (\boldsymbol{\xi}_R, t = \Gamma(\boldsymbol{\xi}_R))$ will be used for a point on Γ pertaining to $\boldsymbol{\xi}_R$, i.e., to the fixed source-receiver pair (S_R, G_R).

- At any specified point S, on the measurement surface Σ_M, explodes a point source with a certain source signal. Its time dependence can be described by the analytic delta function $\Delta(t)$ (REF??). In practice, of course, the analytic delta function has to be replaced by its high-frequency part convolved with some real source pulse that may have a limited bandwidth. The effects of limited bandwidth, however, do not influence the analysis carried out in this paper and need not be considered here.

- Moreover, we assume reproducible point sources of unit strength and an omnidirectional radiation pattern. We also neglect the transmission loss due to interfaces in the overburden. In addition, all other factors affecting the seismic amplitudes apart from geometrical spreading are assumed to be negligible or have been corrected for.

Under the above assumptions, zero-order ray-theory form provides the following description of a primary reflected elementary wave. For every source-receiver pair (S, G), with $\boldsymbol{\xi}$ in A, the reflection event at the receiver is described in analytic form by

$$K_\Gamma(\boldsymbol{\xi}, t) = \frac{\mathcal{R}(M_R)}{\mathcal{L}}\, \Delta(t - \Gamma(\boldsymbol{\xi}))\ . \tag{1}$$

In the above formula, the amplitude factors $\mathcal{R}(M_R)$ and $\mathcal{L}$ are the plane-wave reflection coefficient at M_R and the geometrical-spreading factor pertaining to the specular reflection ray SM_RG. Note again that each point N on Γ defines exactly one point M_R on Σ.

We see that $K_\Gamma(\boldsymbol{\xi}, t)$ is aligned along the reflection-traveltime surface Γ as defined above. We may say that $K_\Gamma(\boldsymbol{\xi}, t)$ is the image of the reflector Σ at the reflection-traveltime surface Γ in the time domain. In other words, the image $K_\Gamma(\boldsymbol{\xi}, t)$ describes what we can observe about the reflector in the recorded reflected wavefield.

We next introduce the function $I_\Sigma(\mathrm{x}, z)$, which is aligned along the target reflector Σ. For each x in E and all real z, the function $I_\Sigma(\mathrm{x}, z)$ is defined by

$$I_\Sigma(\mathrm{x}, z) = \mathcal{R}(M)\, \Delta(z - \Sigma(\mathrm{x}))\ , \tag{2}$$

The function $I_\Sigma(\mathrm{x}, z)$ can be conceived as the result of a true-amplitude depth migration (Schleicher et al., 1993) of the time-domain reflector image $K_\Gamma(\boldsymbol{\xi}, t)$. In other words, the function $I_\Sigma(\mathrm{x}, z)$ describes the depth-domain true-amplitude reflector image of the target reflector Σ in the depth domain. Analogously, the function $K_\Gamma(\boldsymbol{\xi}, t)$ can be conceived as the result of a true-amplitude demigration (Hubral et al., 1996) of the depth-domain reflector image $I_\Sigma(\mathrm{x}, t)$.

Note that the function $I_\Sigma(\mathrm{x}, z)$ is the complex version of the *singular function of the reflector* as introduced by Bleistein (1987). It is defined here, however, in a true-amplitude sense, i.e., with the varying reflection coefficient along the reflector as its amplitude. Moreover, in the same way as expression (2) is referred to as the analytic singular function of the reflector, we can interpret (1) as the analytic singular function of the reflection traveltime surface.

Due to the above observations, we may state that each point N on Γ is associated to a single point M_R on Σ and each point M on Σ is associated to a single point N_R on Γ. The relation between these points is established by the respective specular reflection rays SM_RG and S_RMG_R. Thus, the points on Σ and Γ enjoy a *duality* relationship. The two fundamental singular functions $I_\Sigma(\mathrm{x}, t)$ and $K_\Gamma(\boldsymbol{\xi}, t)$ can, correspondingly, be called *dual* functions of each other. Tygel et al. (1995) have shown that there exists an even closer relationship in mathematical terms between both functions involving various dualities.

DIFFRACTION TRAVELTIMES AND SPATIAL ISOCHRONES

For arbitrary vector parameters $\boldsymbol{\xi}$ in A and arbitrary subsurface points $M = (\mathrm{x}, z)$, we introduce the *diffraction traveltime* surface

$$t = \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z) = T(S(\boldsymbol{\xi}), M) + T(G(\boldsymbol{\xi}), M) \;, \tag{3}$$

namely the sum of traveltimes from the source and receiver pair specified by $\boldsymbol{\xi}$ to the subsurface point M. For a fixed point M, the above formula expresses the traveltimes from the *diffractor point* M to the source- and -receiver pairs specified by varying $\boldsymbol{\xi}$. This explains the adopted terminology.

It is also useful to consider the restriction of the diffraction traveltime function to diffraction points on the reflector Σ. We then introduce the traveltime surface

$$t = \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x}) = \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, \Sigma(\mathrm{x})) \;. \tag{4}$$

In view of the above definitions, the reflection traveltime surface $t = \Gamma(\boldsymbol{\xi})$ of the given refletor Σ can be recast as

$$t = \Gamma(\boldsymbol{\xi}) = \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x}_R(\boldsymbol{\xi})) = \mathcal{T}_D(\boldsymbol{\xi}, \boldsymbol{\xi}_R, \Sigma(\mathrm{x}_R)) \;, \tag{5}$$

where the horizontal vector coordinate $\mathrm{x}_R = \mathrm{x}_R(\boldsymbol{\xi})$ locates the the reflection point $M_R = (\mathrm{x}_R, \Sigma(\mathrm{x}_R)$ on Σ determined by the source- and -recever pair specified by $\boldsymbol{\xi}$.

We next consider the spatial counterparts of the traveltime functions defined above. For any x oin E and arbitrary points $N = (\boldsymbol{\xi}, t)$ in record space, we introduce the *isochrone* function $z = \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, t)$. For any fixed point $N = (\boldsymbol{\xi}, t)$, this surface is the locus of points M_I for which the diffraction traveltimes to the source- and -receiver pairs specified by varying $\boldsymbol{\xi}$ equal the given traveltime t. This is the reason for the terminology isochrone (equal time) function. In symbols, points $M_I = (\mathrm{x}, \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, t)$ on the isochrone are implicitly defined by the condition[2]

$$\mathcal{T}_D(\boldsymbol{\xi}, M_I) = T(S(\boldsymbol{\xi}), M_I) + T(G(\boldsymbol{\xi}), M_I) = t \;. \tag{6}$$

Analogously as before, we find it useful to consider the restriction of the above-defined isochrone function to points N on the traveltime surface Γ of the reflector Σ. This gives rise to the function

$$z = \mathcal{Z}_{IR}(\mathrm{x}, \boldsymbol{\xi}) = \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, t = \Gamma(\boldsymbol{\xi})) \;. \tag{7}$$

[2]We assume, throughout this paper, that for all $\mathbf{x}$ in E, for all $\boldsymbol{\xi}$ in A and for all t under consideration, isochrones $z = \mathcal{Z}_I(\mathbf{x}, \boldsymbol{\xi}, t))$ defined by condition (6) exist asunique, smooth functions. This, of course should impose restrictions on the shape of the reflector, as well as on the measuring configuration. These matters will not be addressed in the present work.

Points $M_I = (\mathrm{x}, z = \mathcal{Z}_{IR}(\mathrm{x}, \boldsymbol{\xi}))$ on the above surface are implicitly defined by the condition

$$\mathcal{T}_D(\boldsymbol{\xi}, M_I) = T(S(\boldsymbol{\xi}), M_I) + T(G(\boldsymbol{\xi}), M_I) = \Gamma(\boldsymbol{\xi}) \ , \tag{8}$$

A final observation is that the reflector function $z = \Sigma(\mathrm{x})$ can be recast as a restriction of the above isochrone functions, namely

$$z = \Sigma(\mathrm{x}) = \mathcal{Z}_{IR}(\mathrm{x}, \boldsymbol{\xi}_R) = \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}_R, \Gamma(\boldsymbol{\xi}_R)) \ , \tag{9}$$

where $\boldsymbol{\xi}_R = \boldsymbol{\xi}_R(\mathrm{x})$ is the vector parameter that specifies the source- and -receiver pair S_R and G_R for which the two ray segments $S_R M and M G_R)$, with $M = (\mathrm{x}, \Sigma(\mathrm{x})$ constitute a reflection ray.

As described in Tygel et al. (1995) and briefly reviewed below, diffraction traveltimes $t = \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x})$, for fixed x, and isochrone surfaces $z = \mathcal{Z}_{IR}(\mathrm{x}, \boldsymbol{\xi})$ for fixed $\boldsymbol{\xi}$ are connected by duality relationships. The diffraction-traveltime surface for a point M on Σ is tangent to Γ at a point N (supposed unique). Correspondingly, the isochrone surface for N on Γ is tangent to Σ at M. Also, further relationships between the dips and curvatures of Σ and Γ in M and N, respectively, can be established.

DUALITY THEOREMS

For a given configuration of source- and -receiver pairs distributed along a measurement surface, the first and second duality theorems of Tygel et al. (1995) provide fundamental geometrical relationships between a reflector Σ and its corresponding primary reflection traveltime surface Γ. As these relationships are crucial to the derivation of practically all the results presented in this paper, we find it convenient to briefly state and comment them in this section.

The duality theorems to be stated below relate tangents and normals (first-order derivative) and curvatures (second-order derivative) properties concerning the *reflector* $\Sigma : z = \Sigma(\mathrm{x})$, with x in E and its *reflection traveltime surface* $\Gamma : t = \Gamma(\boldsymbol{\xi})$, with $\boldsymbol{\xi}$ in A, called throughout the *fundamental dual surfaces*. The relationships between the fundamenal dual surfaces are given in terms of *auxiliary surfaces*, namely the *isochrones* $\Sigma_N : z = \mathcal{Z}_I(\mathrm{x}, N)$ with N in Γ and the *diffraction traveltimes* $\Gamma_M : t = \mathcal{T}_D(\boldsymbol{\xi}, M)$ with M in Σ.

To state the duality theorems, we make use of all traveltime and spatial functions introduced in the text. The traveltime surfaces were (a) the diffraction traveltime function $t = \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z)$, defined for arbitrary vector paramters $\boldsymbol{\xi}$ and points (x, z); (b) its restriction $t = \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x}) = \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, \Sigma(\mathrm{x})$ to points on the reflector $(\mathrm{x}, z = \Sigma(\mathrm{x}))$ and (c) the traveltime $t = \Gamma(\boldsymbol{\xi})$ of the reflector Σ. As explained in the text, this function could also be interpreted as a restriction of the of the previous traveltime functions. The spatial counterparts of the preceding traveltime functions were (a) the isochrone function $z = \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, t)$, defined for arbitrary horizontal vectors x and points $(\boldsymbol{\xi}, t)$; (b) its restriction $z = Z_{IR}(\mathrm{x}, \boldsymbol{\xi}) = \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, \Gamma(\boldsymbol{\xi}))$ to points on the reflection traveltime surface $(\boldsymbol{\xi}, \Gamma(\boldsymbol{\xi}))$ and (c) the reflector $z = \Sigma(\mathrm{x})$, which could also be iterpreted as a further restriction of the preceding spatial functions. With the above definitions, we are ready to state the duality theorems.

First duality theorem:

(a) For any given point M at the reflector Σ, its corresponding diffraction traveltime surface Γ_M is *tangent* to the reflection traveltime Γ at a unique point N, called the *dual point* of M;

(b) For any given point N at the reflection traveltime Γ, its corresponding isochrone surface Σ_N is *tangent* to the reflector Σ at a unique point M, called the *dual point* of N. The points M and N in (a) and (b) are called *dual* of each other;

(c) For fixed $\boldsymbol{\xi}$ in A and varying points $M = (\mathrm{x}, z)$, as well as for fixed x in E and varying points $N = (\boldsymbol{\xi}, t)$, define the partial-derivative functions

$$m_D(\boldsymbol{\xi}, M) = \partial_z \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z) \quad \text{and} \quad \mathrm{n_I}(\mathrm{x}, \mathrm{N}) = \partial_\mathrm{t} \mathcal{Z}_\mathrm{I}(\mathrm{x}, \boldsymbol{\xi}, \mathrm{t}) \ . \tag{10}$$

For any dual points $M = M(\mathrm{x}_M)$ in Σ and $N = N(\boldsymbol{\xi}_N)$ in Γ, we have

$$m_D(\boldsymbol{\xi}_N, M) \cdot n_I(\mathrm{x}_M, N) = 1 \ . \tag{11}$$

(d) For the dual points $M = M(\mathrm{x}_M)$ in Σ nad $N = N(\boldsymbol{\xi}_N)$ in Γ. Let S_N and G_N denote the source and receiver points specified by $\boldsymbol{\xi}_N$. Considering the reflection ray $S_N M G_N$, let α_R be the the angle the incident ray $S_N M$ makes with the normal of Σ at M, let be the ange between the tangent to Σ at M and the z-axis and let v_R be the velocity of the medium just above M. Finally, let θ_R denote the angle the normal to Γ makes with the t-axis at N. Using the same notation as above, the following results are valid

$$m_D = \frac{2\cos\alpha_R \cos\beta_R}{v_R} \ , \tag{12}$$

$$\partial_\eta \mathcal{T}_D(\boldsymbol{\xi}_N, M) = \frac{m_D}{\cos\beta_R} = \frac{2\cos\alpha_R}{v_R} \ , \tag{13}$$

and

$$\partial_\mu \mathcal{Z}_I(\mathrm{x}_M, N) = \frac{n_I}{\cos\theta_R} = \frac{v_R}{2\cos\alpha_R \cos\beta_R \cos\theta_R} \ . \tag{14}$$

Second duality theorem:

For arbitrary $\boldsymbol{\xi}$ and $M = (\mathrm{x}, z)$ introduce the $\boldsymbol{\xi}$-Hessian matrices

$$\underset{\sim}{\boldsymbol{H}}_D(\boldsymbol{\xi}, \mathrm{x}, z) = \left(\frac{\partial^2 \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z)}{\partial \xi_i \partial \xi_j}\right), \quad \underset{\sim}{\boldsymbol{\Gamma}}(\boldsymbol{\xi}) = \left(\frac{\partial^2 \Gamma(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\right) \text{ and } \underset{\sim}{\boldsymbol{H}}_{DR}(\boldsymbol{\xi}, \mathrm{x}) = \left(\frac{\partial^2 \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x})}{\partial \xi_i \partial \xi_j}\right). \tag{15}$$

In the same way, for arbitrary x and $N = (\boldsymbol{\xi}, t)$, introduce the x-Hessian matrices matrices

$$\underset{\sim}{\boldsymbol{\mathcal{Z}}}_I(\mathrm{x}, \boldsymbol{\xi}, t) = \left(\frac{\partial^2 \mathcal{Z}_I(\mathrm{x}, \boldsymbol{\xi}, t)}{\partial x_i \partial x_j}\right), \quad \underset{\sim}{\boldsymbol{\Sigma}}(\mathrm{x}) = \left(\frac{\partial^2 \Sigma(\mathrm{x})}{\partial x_i \partial x_j}\right) \text{ and } \mathcal{Z}_{IR}(\mathrm{x}, \boldsymbol{\xi}) = \left(\frac{\partial^2 \mathcal{Z}_{IR}(\boldsymbol{\xi}, \mathrm{x})}{\partial x_i \partial x_j}\right). \tag{16}$$

Introduce finally the mixed-derivative matrices

$$\underset{\sim}{\boldsymbol{\Lambda}}_{DR}(\boldsymbol{\xi}, \mathrm{x}, z) = \left(\frac{\partial^2 \mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x})}{\partial \xi_i \partial x_j}\right) \quad \text{and} \quad \underset{\sim}{\boldsymbol{H}}_{\mathrm{B}}(\boldsymbol{\xi}, \mathrm{x}, \mathrm{z}) = \begin{pmatrix} \nabla_{\mathbf{x}} \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z) \\ \partial_{\xi_1} \nabla \mathcal{T}_D(\boldsymbol{\xi}, \mathrm{x}, z) \\ \partial_{\xi_2} \nabla \mathcal{T}_D)\boldsymbol{\xi}, \mathrm{x}, z) \end{pmatrix}. \tag{17}$$

For any pair of dual points $M = (\mathrm{x}, z)$ in Σ and $N = (\boldsymbol{\xi}, t)$ in Γ, let the above matrices be evaluated at the respective coordinates and points. Introduce further the simplifying notations for the obtained matrices by leaving out their arguments. Then the following relationships are valid

$$\underset{\sim}{\boldsymbol{H}}_D - \underset{\sim}{\boldsymbol{\Gamma}} = \underset{\sim}{\boldsymbol{\Lambda}}_{DR}\, \underset{\sim}{\boldsymbol{H}}_{DR}^{-1}\, \underset{\sim}{\boldsymbol{\Lambda}}_{DR}^T\,, \tag{18}$$

$$m_D\, (\underset{\sim}{\boldsymbol{\mathcal{Z}}}_I - \underset{\sim}{\boldsymbol{\Sigma}}) = -\, \underset{\sim}{\boldsymbol{H}}_{DR}\,, \tag{19}$$

and

$$m_D\, \underset{\sim}{\boldsymbol{\mathcal{Z}}}_{IR} = \underset{\sim}{\boldsymbol{\Lambda}}_{DR}\, \underset{\sim}{\boldsymbol{H}}_{DR}\, \underset{\sim}{\boldsymbol{\Lambda}}_{DR}^T\,. \tag{20}$$

A useful relationship can also be obtained after elimination of $\underset{\sim}{\boldsymbol{H}}_{DR}$ in equations (18) and (19), namely

$$m_D\, (\underset{\sim}{\boldsymbol{H}}_D - \underset{\sim}{\boldsymbol{\Gamma}}) = -\, \underset{\sim}{\boldsymbol{\Lambda}}_{DR}\, (\underset{\sim}{\boldsymbol{\mathcal{Z}}}_I - \underset{\sim}{\boldsymbol{\Sigma}})^{-1}\, \underset{\sim}{\boldsymbol{\Lambda}}_{DR}^T\,, \tag{21}$$

Moreover, we have the additional relationship

$$\det \underset{\sim}{\boldsymbol{\Lambda}}_{DR} = \frac{h_B}{m_D}\,, \tag{22}$$

where

$$h_B = \det \underset{\sim}{\boldsymbol{H}}_B \tag{23}$$

is the so-called Beylkin determinant (Beylkin, 1985, Bleistein, 1987).

It is to be reminded that the first duality theorem and equations (18), (19) and (21) have been proved in Tygel et al. (1995). The new equation (21) is proved in Appendix A. With the inclusion of the this last equation, we call the above results the duality theorems in their complete form.

THE KIRCHHOFF-HELMHOLTZ INTEGRAL PAIR

The Kirchhoff-Helmholtz (KH) modeling integral, called from now on the forward KH integral, asymptotically computes the singular function $K_\Gamma(\boldsymbol{\xi}, t)$ of the reflection-traveltime surface Γ. Input to this calculation is the location of the reflector Σ, the velocity distribution in the depth domain, and the values of the reflection coefficient $\mathcal{R}(M)$ along Σ.

In a completely analogous way, the inverse KH integral to be defined below, asymptotically computes the singular function $I_\Sigma(\mathbf{x}, z)$ of the reflector Σ. Input to this calculation is the location of the reflection-traveltime surface Γ, the velocity distribution in the depth domain, and the wavefield amplitude $\mathcal{R}(M_R)/\mathcal{L}$ along Γ.

In this section, we discuss both the forward and inverse KH integrals as well as their basic dual properties. The weight functions, which are also presented here, will be derived in the next section.

Under the assumptions stated in the previous section, the forward KH integral can be written as an integral along the reflector Σ in the form (Frazer and Sen, 1985)

$$K(\boldsymbol{\xi}, t) = \frac{1}{4\pi} \int d\Sigma \; \mathcal{W}_K(\boldsymbol{\xi}, M) \; \mathcal{R}(M) \; \partial_\eta \Delta(t - \mathcal{T}_D(\boldsymbol{\xi}, M)) \; , \tag{24}$$

where $K(\boldsymbol{\xi}, t)$ is the modeled elementary wave at the receiver $G(\boldsymbol{\xi})$. Also, ∂_η denotes the partial derivative in the direction of the normal to the surface Σ at M. Under the above-mentioned assumption that transmission losses in the overburden can be neglected, the weight function is given by

$$\mathcal{W}_K(\boldsymbol{\xi}, M) = \frac{1}{\mathcal{L}_S \, \mathcal{L}_G} \; , \tag{25}$$

where $\mathcal{L}_S$ and $\mathcal{L}_G$ denote the geometrical-spreading factors along the two ray branches from the source S to the point M and from there to the receiver G (see Figure 1).

Let us now investigate integral (24) more closely in order to better understand it geometrically. This will help us to set up an analogous integral for its inversion. For the following discussion, we refer to Figure 1.

We start by considering a certain, fixed value $\overline{\boldsymbol{\xi}}$ where we want to compute the reflected wave as a function of time. We recall that $\overline{\boldsymbol{\xi}}$ defines a certain, fixed source-receiver pair, which we denote by $(\overline{S}, \overline{G})$. We denote by M_R the (supposedly unique) reflection point on Σ which correspond to the source-receiver pair $(\overline{S}, \overline{G})$. The point M_R, on its turn, defines a diffractin traveltime surface that is tangent to the reflection traveltime surface Γ on a (supposedly unique) point. This point, denoted by $\overline{N}$ is called the dual to point M_R. We observe that for each point M on the reflector, integral (24) contributes to the final response $K(\overline{\boldsymbol{\xi}}, t)$ at a single point $Q = (\overline{\boldsymbol{\xi}}, t = \mathcal{T}_D(\overline{\boldsymbol{\xi}}, M))$, where $\mathcal{T}_D$ is defined in equation (3) as the sum of traveltimes along the rays $\overline{S}M$

and $M\overline{G}$. In other words, Q is the point where the diffraction traveltime surface for point M, $t = \mathcal{T}_D(\boldsymbol{\xi}, M)$, cuts the vertical line at $\overline{\boldsymbol{\xi}}$ (see Figure 1). We remind that the surface $t = \mathcal{T}_D(\boldsymbol{\xi}, M)$ is given by all traveltimes along the rays from any source-receiver pair (S, G) to point M (dashed rays in Figure 1). The point Q will fall onto Γ, i.e., it will coincide with point $\overline{N}$, the dual point to $\overline{N}$, when M coincides with M_R. At $\overline{N}$, the diffraction traveltime surface of M_R, $t = \mathcal{T}_D(\boldsymbol{\xi}, M_R)$, is tangent to Γ. Due to our assumption that the reflector Σ is continuous and smooth, we thus have a stationary situation at $\overline{N}$, which means that the main contribution of integral (24) will be observed at that point. In other words, the forward KH integral (24) transforms the singular function of reflector Σ into its image at Γ. The weight function $\mathcal{W}_K(\boldsymbol{\xi}, M)$ serves to perform this transformation in a dynamically correct way, i.e., yielding the correct wave amplitude at $\overline{N}$.

To set up a completely analogous integral that achieves exactly the inverse task, namely to reconstruct the singular function of the reflector Σ from its image at Γ, we only have to substitute in the above integral all points and surfaces by their respective duals. This is geometrically described with the help of Figure 2.

The new integral to be set up has to consist of an integration along the reflection-traveltime surface Γ instead of the reflector Σ. As before, we consider the integration result at a certain, fixed coordinate $\overline{\boldsymbol{x}}$ which defines a point $\overline{M}$ on the reflector Σ. The point MM determines a (supposedly unique) dual point $N_R = N_R$ on the reflection traveltime surface Γ. The isochrone specified by the point N_R will be tangent to the reflector Σ at the (supposedly unique) point $\overline{M}$. The points $\overline{M}$ and N_R are said to be dual points. For each point N on Γ, the new integral has to contribute to the final result $I(\overline{\boldsymbol{x}}, z)$ at a certain point P. This point P must be located at the position where the isochrone of N, $z = \mathcal{Z}(\mathrm{x}, N)$ cuts the vertical line at $\overline{\boldsymbol{x}}$. In symbols, $P = (\overline{\boldsymbol{x}}, z = \mathcal{Z}(\overline{\boldsymbol{x}}, N))$. The point P will fall ont Σ, i.e., it will coincide with $\overline{M}$, when N coincides with N_R, the dual point of $\overline{M}$. At $\overline{M}$, the isochrone $z = \mathcal{Z}(\mathrm{x}, N_R)$ is be tangent to Σ. Due to our above assumption of a smooth reflector and uniqueness of dual points, we have again the situation of an isolated singularity at $\overline{M}$, which means that the main contribution of the new integral will be observed at $\overline{M}$. In this way, we have geometrically constructed a transformation of the reflection-traveltime function Σ into the reflector Σ. A free weight function will be included into the integral in order to assure that also this inverse transformation can be performed in a dynamically correct way, too, i.e., to correctly reconstruct the varying reflection coefficient along the reflector Σ.

Transforming the above observations into mathematical terminology in full correspondence to the forward KH integral, we can now set up the following inverse KH integral,

$$I(\mathrm{x}, z) = -\frac{1}{4\pi} \int d\Gamma\, \mathcal{W}_I(\mathrm{x}, N)\, \frac{\mathcal{R}(M_R)}{\mathcal{L}}\, \partial_\mu \Delta(z - \mathcal{Z}_I(\mathrm{x}, N))\,. \tag{26}$$

where $I(\mathrm{x}, z)$ is the final imaging result. In this formula, ∂_μ denotes, correspondingly to ∂_η above, the partial derivative in the direction of the normal to the traveltime surface Γ at N. We recall that M_R is the specular reflection point on the reflector pertaining to the source-receiver pair (S, G) defined by $\boldsymbol{\xi}$. From the analysis to be carried out below, it will become clear that the weight function can be represented as

$$\mathcal{W}_I(\mathbf{x}, N) = \frac{h_B \; v^2 \; \cos^2\theta}{\cos^2\alpha} \mathcal{L}_S \mathcal{L}_G \; , \tag{27}$$

where θ represents the "local dip angle" of the reflection-traveltime surface Γ (i.e., the angle the normal to Γ at N makes with the vertical t-axis), and α denotes the incidence angle the incoming ray-branch slowness vector makes with the isochrone normal at P (see Figure 2). Moreover, h_B is the modulus of the Beylkin determinant (Beylkin, 1985; Bleistein, 1987). All these quantities are computed for the actual point N on Γ.

In mathematical terms, the two stationary situations mentioned above relate to the following statements for the asymptotic integral results. As is well known (see, for example, Bleistein, 1984); Tygel et al., 1994) the Kirchhoff-Helmholtz integral (24) can be evaluated in the high-frequency approximation, such that, in an asymptotic sense, it equals the zero-order ray-theoretical expression, viz.,

$$K(\boldsymbol{\xi}, t) \approx K_\Gamma(\boldsymbol{\xi}, t) = \frac{\mathcal{R}(M_R)}{\mathcal{L}(\boldsymbol{\xi})} \; \Delta(t - \Gamma(\boldsymbol{\xi})) \; . \tag{28}$$

As indicated above, we shall show that correspondingly, the evaluation of integral (26) in high-frequency approximation yields, in an asymptotic sense,

$$I(\mathbf{x}, z) \approx I_\Sigma(\mathbf{x}, z) = \mathcal{R}(M) \; \Delta(z - \Sigma(\mathbf{x})) \; , \tag{29}$$

i.e., the (complex) singular function of the reflector as defined above. This means that integral (26) is the inverse to the forward KH integral (24), or, in other words, integrals (24) and (26) form a transform pair between the depth-domain image $I_\Sigma(\mathbf{x}, z)$ of the target reflector and its time-domain image $K_\Gamma(\boldsymbol{\xi}, t)$ in multi-coverage reflection data.

ASYMPTOTIC EVALUATION OF THE INTEGRALS

In the same way as the forward KH integral, also its inverse defined above admits simple asymptotic evaluation, irrespective of the specific form of the weight function, which will, for the time being, be left unspecified. In the following section, we will compare the asymptotic evaluations of both KH integrals to confirm the correct duality of the transformation pair given by equations (24) and (26). The kinematic part of the analysis will show that an inverse KH integral of the form defined in equation (26) exists, and the dynamics of both integrals will, in fact, determine the adequate form of the necessary weight function in integral (26).

Asymptotic evaluation of the forward KH integral

In high-frequency approximation, the leading term of the asymptotic evaluation of the forward KH integral (24) is given by

$$K_{as}(\boldsymbol{\xi}, t) = \frac{\mathcal{R}(M_R)}{2\mathcal{L}_S\,\mathcal{L}_G\cos\beta_R}\,\frac{e^{i\pi\,(2-\sigma)/4}}{\sqrt{|H|}}\,\partial_\eta \mathcal{T}_D(\boldsymbol{\xi}, M_R)\,\Delta(t-\Gamma(\boldsymbol{\xi}))\ , \tag{30}$$

where $M_R = M(\boldsymbol{x}_R)$ is the (unique) specular reflection point on the target reflector Σ. It pertains to the source-receiver pair specified by the given configuration vector

parameter $\boldsymbol{\xi}$. Here, $\boldsymbol{x}_R = \boldsymbol{x}_R(\boldsymbol{\xi})$ is the stationary point of integral (24). Also, β_R is the reflector dip at M_R. Finally, σ and H are the signature and the determinant, respectively, of the Hessian matrix of the traveltime,

$$\underset{\sim}{\boldsymbol{H}}_{DR}(\boldsymbol{\xi}, \mathrm{x}) = \left(\frac{\partial^2\mathcal{T}_{DR}(\boldsymbol{\xi}, \mathrm{x})}{\partial x_i \partial x_j}\ ,\right) \tag{31}$$

evaluated at $(\boldsymbol{\xi}, \mathrm{x}_R)$. The projection of this matrix into the tangent plane to the reflector Σ at M_R is referred to as the Fresnel matrix, because it was shown in Hubral et al. (1992) that it is this projected matrix which defines the size of the Fresnel zone at M_R.

Using the well-known fact that the high-frequency evaluation of the forward KH integral (24) yields the zero-order ray-theoretical expression (1) (Bleistein, 1984), one obtains the following decomposition formula for the geometrical-spreading factor $\mathcal{L}$ along ray SM_RG

$$\mathcal{L} = \frac{\mathcal{L}_S\,\mathcal{L}_G v_R\cos\beta_R}{\cos\alpha_R}\,e^{i\pi\,(\sigma-2)/4}\sqrt{|H|}\ , \tag{32}$$

where we have used that, at the stationary point M_R,

$$\partial_z\mathcal{T}_D(\boldsymbol{\xi}, M_R) = \frac{2\cos\alpha_R\,\cos\beta_R}{v_R}\ , \tag{33}$$

from which

$$\partial_\eta\mathcal{T}_D(\boldsymbol{\xi}, M_R) = \frac{1}{\cos\beta_R}\partial_z\mathcal{T}_D(\boldsymbol{\xi}, M_R) = \frac{2\cos\alpha_R}{v_R}\ . \tag{34}$$

Here, v_R is the local velocity and α_R is the specular reflection angle at M_R. Introducing the *Fresnel geometrical-spreading factor* (Tygel et al., 1994)

$$\mathcal{L}_F = \frac{\cos\alpha_R}{v_R\cos\beta_R}\frac{e^{i\pi\,(2-\sigma)/4}}{\sqrt{|H|}}\ , \tag{35}$$

we find the following formula for the geometrical-spreading decomposition

$$\mathcal{L} = \frac{\mathcal{L}_S\,\mathcal{L}_G}{\mathcal{L}_F}\ . \tag{36}$$

This formula will be of further use in the evaluation of the inverse KH integral (26).

Asymptotic evaluation of the inverse KH integral

In the same way, the asymptotic evaluation of the inverse KH integral (26) yields

$$I_{as}(\mathbf{x}, z) = \frac{\mathcal{R}(M)}{\mathcal{L}} \frac{\mathcal{W}_I(\mathbf{x}, N_R)}{2\cos\theta_R} \frac{e^{-i\pi\,(2+\gamma)/4}}{\sqrt{|Z|}} \,\partial_\mu \mathcal{Z}_I(\mathbf{x}, N_R)\,\Delta(z - \Sigma(\mathbf{x}))\;, \tag{37}$$

where $N_R = N(\boldsymbol{\xi}_R)$ specifies the source-receiver pair (S_R, G_R) for which the reflector point $M(\mathbf{x})$ is a specular reflection (stationary) point. Here, $\boldsymbol{\xi}_R = \boldsymbol{\xi}_R(\mathbf{x})$ is the stationary point of integral (26). In other words, N_R, on the reflection-traveltime surface Γ, is the dual point to the point $M(\mathbf{x})$ on the target reflector Σ in the vicinity of which the integral (26) is calculated. Finally, γ and Z are the signature and the determinant, respectively, of the isochrone Hessian matrix

$$\underset{\sim}{\boldsymbol{Z}}_{IR}(\mathbf{x}, \boldsymbol{\xi}) = \left(\frac{\partial^2 \mathcal{Z}_{IR}(\mathbf{x}, \boldsymbol{\xi})}{\partial\xi_i \partial\xi_j}\right)\;, \tag{38}$$

evaluated at $(\mathbf{x}, \boldsymbol{\xi}_R)$, and θ_R is the local reflection-traveltime dip at N_R.

As shown in the Appendix, we have that

$$\partial_\mu \mathcal{Z}_I(\boldsymbol{\xi}, N_R) = \frac{v_R}{2\cos\alpha_R \cos\beta_R \cos\theta_R}\;, \tag{39}$$

and also

$$\frac{e^{-i\pi\,(2+\gamma)/4}}{\sqrt{|Z|}} = \frac{4\cos^3\alpha_R \cos\beta_R}{v_R^3 h_B \mathcal{L}_F}\;. \tag{40}$$

Substitutution of the above expressions into equation (37) together with the use of the geometrical-spreading decomposition formula (32), leads to

$$I_{as}(\mathbf{x}, z) = \mathcal{R}(M)\;\mathcal{W}_I(\mathbf{x}, N_R)\;\frac{\cos^2\alpha_R}{h_B\, v_R^2\,\cos^2\theta_R \mathcal{L}_S\,\mathcal{L}_G}\;\Delta(z - \Sigma(\mathbf{x}))\;. \tag{41}$$

We observe that equation (38) kinematically reconstructs the singular function (29) of the reflector Σ. To also achieve correct dynamic reconstruction, we have to choose the weight function $\mathcal{W}_I(\mathbf{x}, N_R)$ such that the amplitude factor in equation (41) equals $\mathcal{R}(M)$. This determines the sought-for weight function as

$$\mathcal{W}_I(\mathbf{x}, N_R) = \frac{h_B\;v_R^2\;\cos^2\theta_R}{\cos^2\alpha_R}\mathcal{L}_S\mathcal{L}_G\;, \tag{42}$$

Observing that there is no quantity involved that depends on the reflector, we can generalize this weight function to any arbitrary point N. This final weight function of the inverse Kirchhoff integral (26) is the one stated in equation (27).

CONCLUSIONS

We have presented a completely analogous inverse to the well-known forward Kirchhoff-Helmholtz integral. Just as the forward Kirchhoff-Helmholtz integral can be conceived as a superposition of the elementary responses of all Huygens secondary sources along the reflector, we can conceive its inverse as a superposition of "elementary reflection images" along the reflection-traveltime surface. The new inverse Kirchhoff-Helmholtz integral was constructed using the fundamental dual properties that relate the points and surfaces of the time-domain data space and the depth-domain model space. For instance, in the same way as the Huygens secondary sources can be interpreted mathematically as the source pulse multiplied with the local reflection coefficient at any point on the reflector, the elementary reflection images can be represented by the local pulse of an elementary reflected wave at any point on the reflection-traveltime surface.

The inverse Kirchhoff-Helmholtz integral fills a recently discovered gap which originates from the observation that the conventional Kirchhoff migration integral (Schneider, 1978) is not an inverse to the forward Kirchhoff-Helmholtz integral. In fact, there exists another inverse to migration, namely the Kirchhoff demigration integral (Hubral et al., 1996; Tygel et al., 1996). Although the latter can be used for modeling purposes (Santos et al., 1998), it is not identical to the forward Kirchhoff-Helmholtz integral. Therefore, the Kirchhoff migration integral cannot be the inverse to forward modeling by the Kirchhoff-Helmholtz integral as is conventional wisdom. In this paper, we have shown that there is indeed a different, although related, inverse Kirchhoff-Helmholtz integral.

The proposed inverse Kirchhoff-Helmholtz integral enables the design of a new seismic migration technique that would deserve the name Kirchhoff migration much more than what is up to now associated with this name. The construction of true-amplitude migrated reflector images by the new migration technique can be achieved by the superposition of their elementary reflection images along the reflection-traveltime surface. In this way, the migration can be realized as a weighted stack along the (identified and picked) reflection-time surface instead of the conventional diffraction-time surfaces (that have to be calculated in a macro-velocity model). Of course, to recover the correct reflector position as well as to calculate the weight function, also the new migration technique needs a macro-velocity model.

REFERENCES

Beylkin, G., 1985, Imaging of discontinuities in the inverse scattering problem by inversion of a generalized Radon transform: J. Math. Phys., **26**, no. 1, 99–108.

Bleistein, N., 1984, Mathematics of wave phenomena: Academic Press, New York.

Bleistein, N., 1987, On the imaging of reflectors in the earth: Geophysics, **52**, no. 7, 931–942.

Frazer, L., and Sen, M., 1985, Kirchhoff-Helmholtz reflection seismograms in a laterally inhomogeneous multi-layered elastic medium – I. Theory: Geophys. J.Roy. Astr. Soc., **80**, 121–147.

Hubral, P., Schleicher, J., and Tygel, M., 1992, Three-dimensional paraxial ray properties – Part I. Basic relations: J. Seis. Expl., **1**, no. 3, 265–279.

Hubral, P., Schleicher, J., and Tygel, M., 1996, A unified approach to 3-D seismic reflection imaging – Part I: Basic concepts: Geophysics, **61**, no. 3, 742–758.

Klem-Musatov, K., and Aizenberg, A., 1984, The ray method and the theory of edge waves: Geophys. J.Roy. Astr. Soc., **79**, no. 1, 35–50.

Santos, L., Schleicher, J., Tygel, M., and Hubral, P., 1998, Seismic modeling by demigration: Geophysics.

Schleicher, J., Tygel, M., and Hubral, P., 1993, 3-D true-amplitude finite-offset migration: Geophysics, **58**, no. 8, 1112–1126.

Schneider, W., 1978, Integral formulation for migration in two and three dimensions: Geophysics, **43**, no. 1, 49–76.

Stolt, R., and Benson, A., 1986, Seismic migration – theory and practice:, volume 5 of **Handbook of Geophysical Exploration** Geophysical Press, London – Amsterdam.

Tygel, M., and Ursin, B., 1998, Weak-contrast edge and vertex diffractions in anisotropic elastic media: Wave Motion.

Tygel, M., Schleicher, J., and Hubral, P., 1994, Kirchhoff-Helmholtz theory in modelling and migration: J. Seis. Expl., **3**, no. 3, 203–214.

Tygel, M., Schleicher, J., and Hubral, P., 1995, Dualities between reflectors and reflection-time surfaces: J. Seis. Expl., **4**, no. 2, 123–150.

Tygel, M., Schleicher, J., and Hubral, P., 1996, A unified approach to 3-D seismic reflection imaging – Part II: Theory: Geophysics, **61**, no. 3, 759–775.

APPENDIX A

In this appendix, we provide a derivation of equations (39) and (40), which were crucial for obtaining the weight function (42) of the inverse Kirchhoff-Helmholtz transform. These derivations rely on the use of the duality theorems presented in the text.

A proof of the new result (20) of the second duality theorem will be given in the derivation. Proofs of the remaining duality theorem results can be found in Tygel et al. (1995). To derive equation (39), we first observe that

$$\partial_\mu \mathcal{Z}_I(\mathrm{x}, M_R) = \frac{1}{\cos\theta_R}\, \partial_t \mathcal{Z}_I(\mathrm{x}, M_R) = \frac{n_I}{\cos\theta_R}\ ,. \tag{A-1}$$

In view of equation (11) of the first duality theorem and also upon the use equation (A-8), we readily find

$$\partial_\mu \mathcal{Z}_I(\mathrm{x}, M_R) = \frac{1}{m_D \cos\theta_R} = \frac{v_R}{2\cos\alpha_R \cos\beta_R \cos\theta_R}\ , \tag{A-2}$$

which is equation (39), as required.

We now proceed to prove equation (40), as well as equation (20) of the second duality theorem. Recalling that $M_I(\mathrm{x}, \boldsymbol{\xi}) = (\mathrm{x}, \mathcal{Z}(\mathrm{x}, N(\boldsymbol{\xi})))$ is a point on the isochrone defined by N, we start from the identity (6), which can be alternatively represented as

$$\mathcal{T}_D(\boldsymbol{\xi}, M_I(\mathrm{x}, \boldsymbol{\xi})) = \Gamma(\boldsymbol{\xi})\ . \tag{A-3}$$

Differentiate both sides with respect to ξ_j $(j = 1, 2)$ using the chain rule to obtain

$$\frac{\partial \mathcal{T}_D}{\partial \xi_j} + \partial_z \mathcal{T}_D(\boldsymbol{\xi}, M_I)\, \frac{\partial \mathcal{Z}}{\partial \xi_j} = \frac{\partial \Gamma}{\partial \xi_j}\ . \tag{A-4}$$

At the stationary point N_R, we have

$$\partial \mathcal{Z} / \partial \xi_j = 0\ . \tag{A-5}$$

Therefore, we find the well-known tangency property

$$\frac{\partial \mathcal{T}_D}{\partial \xi_j} = \frac{\partial \Gamma}{\partial \xi_j}\ . \tag{A-6}$$

We next differentiate both sides of equation (A-4) with respect to ξ_i again using the chain rule. At the stationary point N_R, we find upon the use of the stationary condition (A-5)

$$\frac{\partial^2 \mathcal{T}_D}{\partial \xi_i \partial \xi_j} + \partial_z \mathcal{T}_D(\boldsymbol{\xi}, M_R)\, \frac{\partial^2 \mathcal{Z}}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 \Gamma}{\partial \xi_i \partial \xi_j}\ . \tag{A-7}$$

Dividing by

$$m_D = \partial_z \mathcal{T}_D(\boldsymbol{\xi}, M_R)\ , \tag{A-8}$$

we obtain

$$\frac{\partial^2 \mathcal{Z}}{\partial \xi_i \partial \xi_j} = -\frac{1}{m_D}\left[\frac{\partial^2 \mathcal{T}_D}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 \Gamma}{\partial \xi_i \partial \xi_j}\right]\ , \tag{A-9}$$

which, in the notation as introduced before, recovers equation (20)

$$\underset{\sim}{\boldsymbol{Z}} = -\frac{1}{m_D} \left[\underset{\sim}{\boldsymbol{H}}_D - \underset{\sim}{\boldsymbol{\Gamma}} \right] , \tag{A-10}$$

as desired. To find an expression for the determinant of $\underset{\sim}{\boldsymbol{Z}}$, we take the determinant of both sides of equation (A-10). In view of the second duality theorem, equations (18) and (22), we find

$$Z = \det \underset{\sim}{\boldsymbol{Z}} = \frac{h_B^2}{m_D^4} \frac{1}{\det \underset{\sim}{\boldsymbol{H}}} = \frac{h_B^2}{m_D^4 \, H} \tag{A-11}$$

and

$$\gamma = \mathrm{Sgn}\, \underset{\sim}{\boldsymbol{Z}} = -\, \mathrm{Sgn}\, \underset{\sim}{\boldsymbol{H}} = -\sigma \, . \tag{A-12}$$

These two equations can be combined into

$$\frac{e^{-i\pi\,(2+\gamma)/4}}{\sqrt{|Z|}} = \frac{m_D^2 \sqrt{|H|}}{h_B} \, e^{-i\pi(2-\sigma)/4} \, . \tag{A-13}$$

Using the Fresnel geometrical-spreading formula (35), we find

$$\frac{e^{-i\pi\,(2+\gamma)/4}}{\sqrt{|Z|}} = \frac{\cos\alpha_R}{v_R \cos\beta_R} \, \frac{m_D^2}{h_B} \, \mathcal{L}_F \, . \tag{A-14}$$

Use of equation (33)

$$m_D = \partial_z \mathcal{T}_D(\boldsymbol{\xi}, M_R) = \frac{2\cos\alpha_R \, \cos\beta_R}{v_R} \, , \tag{A-15}$$

yields the desired equation (40).

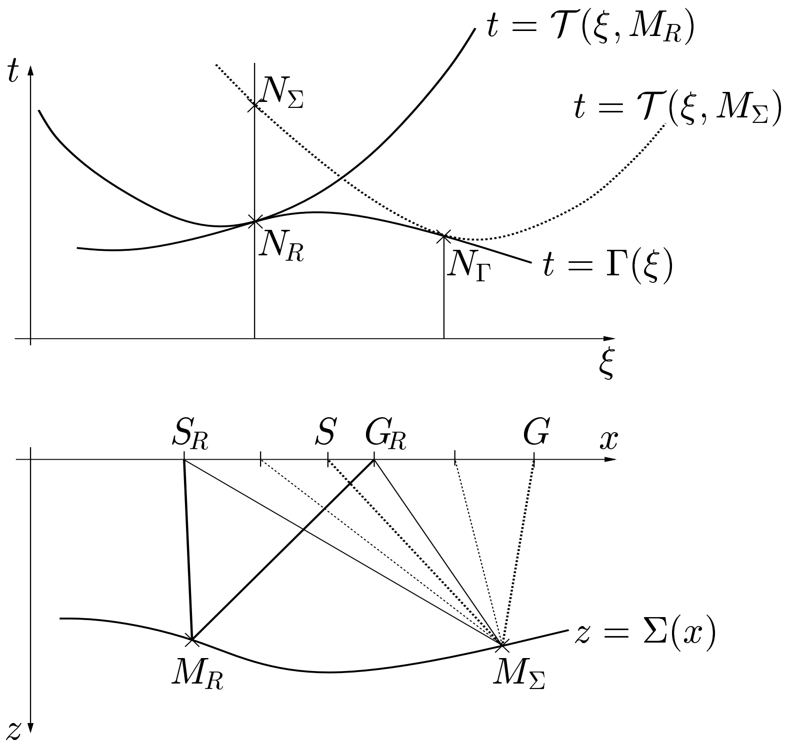

Figure 1: The forward Kirchhoff-Helmholtz integral understood geometrically. For each point M on Σ, the integration contributes to the reflection response computed for $\bar{\boldsymbol{\xi}}$ at the corresponding point $Q = (\bar{\boldsymbol{\xi}}, t = \mathcal{T}_D(\bar{\boldsymbol{\xi}}, M))$. For details see text.

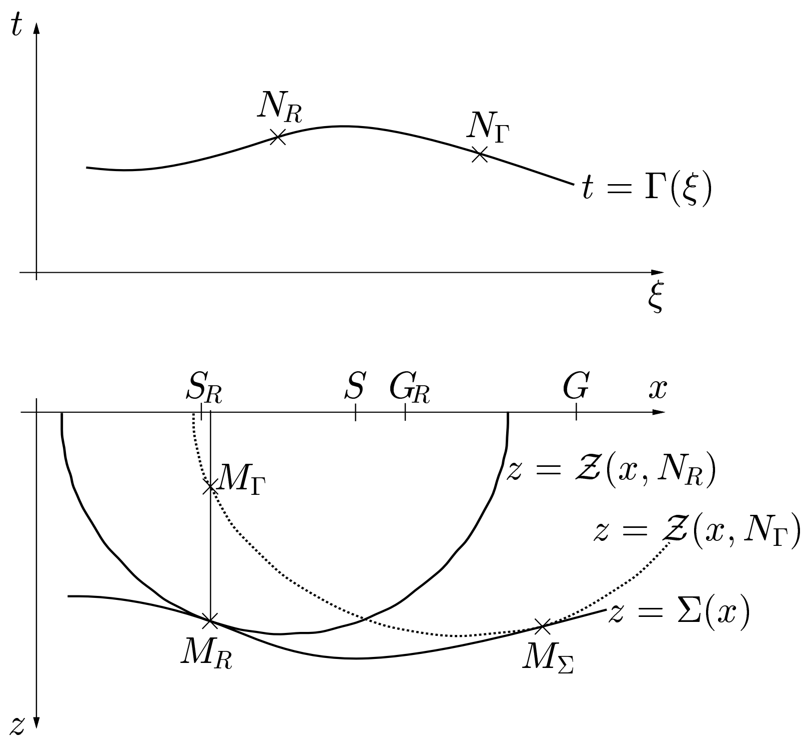

Figure 2: The inverse Kirchhoff-Helmholtz integral understood geometrically. For each point N on Γ, the integration contibutes to the reflector depth image computed for $\overline{\boldsymbol{x}}$ at the corresponding point $P = (\overline{\boldsymbol{x}}, t = \mathcal{Z}(\overline{\boldsymbol{x}}, N))$. For details see text.