Wave Inversion Technology, Report No. 2, pages 207-216

Seismic finite-difference modeling with spatially varying time steps

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keywords: *finite-differences, time integration*

ABSTRACT

Numerical seismic modeling by finite-difference methods usually work with a global time step size. Due to stability considerations the time step size is essentially determined by the highest seismic velocity, i.e., the higher the highest velocity, the smaller the time step needs to be. Therefore, domains of low velocity are temporally oversampled, if large velocity contrasts exist within the numerical grid. Using different time step sizes in different parts of the numerical grid can considerably reduce computational costs.

INTRODUCTION

Seismic modeling becomes increasingly popular for studying wave phenomena in complex structures. With increasing computer power it is now easily possible to solve the elastodynamic equations, whereas in the past merely the acoustic wave equation was solved. Elastic modeling offers much more insight into the mechanisms of wave propagation than acoustic modeling does. However, since in isotropic elastic modeling two wave types are involved spatial and temporal sampling has to cope with both P- and S-waves. Therefore, for elastic modeling the grid spacing needs to be smaller than for acoustic modeling and consequently the time step size has to be smaller. This in turn increases the computational effort.

Topics of interest are for example tube-wave interactions with cracks in order to monitor crack growth in a hydro-frac cycle . In such situations seismic velocities differ greatly, were the steel casing has extremely high P-wave and the mud cake has extremely low S-wave velocities. Another more common situation with large velocity contrasts appears in ordinary field acquisitions. There, S-wave velocities in the weathered zone can be extremely low, whereas the P-waves in the subsurface may have quite high velocities.

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In order to save computational costs one can think of optimizing spatial and temporal sampling according to the local parameters. In domains were high velocities imply long wavelengths the numerical grid can be made more coarse, or were low seismic speeds result in short wavelengths the grid spacing can be made smaller. Such techniques were applied by Jastram and Tessmer (1994) and Moczo (1989) in order to refine the resolution of interfaces. Falk et al. (1996) used such a techni que for better resolution of the borehole and its vicinity. The variation of grid size can be made in sharp jumps or can be made smoothly by mapping functions (Fornberg, 1988). Different temporal sampling in different parts of the numerical grid was introduced by Falk et al. (1997). However, the method is restricted to ratios of time steps between the different domains of 2^n. The new method presented here can handle any positive integer ratio. A further advantage is that the width of the transition zone between domains is constant and does not depend on the time step ratio.

Numerical examples and comparisons with analytic solutions for two halfspaces in juxtaposition demonstrate the accuracy of the method.

SOLUTION OF THE EQUATION OF MOTION

The equations of motion in compact notation reads

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = -\mathbf{L^2}\mathbf{u} + \mathrm{s}, \tag{1}$$

where $\mathbf{u}$ is the displacement vector, s is the source term and the operator $-\mathbf{L^2}$ contains the material parameters and spatial derivatives, e.g. (Kosloff et al., 1989).

The formal solution of equation (1) reads

$$\mathbf{u}(\mathbf{x}, t) = \left[\int_0^t \frac{\sin \mathbf{L}t}{\mathbf{L}} h(t-\tau)d\tau\right] \mathbf{g}(\mathbf{x}), \tag{2}$$

where $h(t)$ is the source time history function and $\mathbf{g}(\mathbf{x})$ is the spatial distribution of sources.

After discretization in time and expanding into a Taylor series the well-known leap-frog second order time integration scheme can be derived from this:

$$u^{n+1} = -u^{n-1} + 2u^n + (\Delta t)^2 \left(-\mathbf{L^2}u^n + s\right). \tag{3}$$

The time stepping is done iteratively from one time level to the next, where the scheme is symmetrically around time level $n\Delta t$.

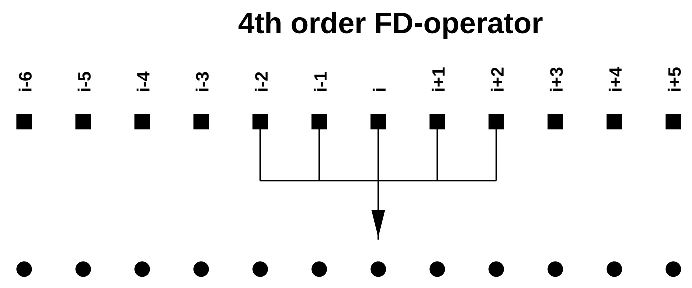

Figure 1: Domain of dependency of a 4th order FD operator.

SCHEME FOR TIME INTEGRATION

Finite-difference derivative operators depend on a few grid points usually located symmetrically around a central point for which the local derivative is to be computed (Figure 1). The number of grid points involved depends on the order of the finite-difference approximation. Difficulties are encountered if the FD operator approaches domain boundaries, since field values, which do not exist at certain time levels, are required by the operator scheme.

If time integration of different time step sizes in two domains is performed, these missing values have to be provided. Within a zone beyond the domain boundary these points can be calculated by the same time integration scheme like in the remainder of the computational domain. The only difference is that the time step size is different. The method for space dependent time integration is based on equation (3) . For the sake of simplicity the procedure is demonstrated for the 1-D acoustic case, though in the examples below it is implemented for the 2-D elastic case. Figure 2 shows the 1-D grid at various time levels were time stepping in the right part is performed three times as often than in the left part. Open circles and diamonds represent the gridpoints where time integration has to be performed at extra time levels. The procedure is as follows: Let us assume a base time step size Δt. We start at time level $n\Delta t$. In Domain 1 and Domain 2 time stepping is done with sizes $3\Delta t$ and $1\Delta t$, respectively. This is done in a conventional way, where values of the previous time level ($(n-3)\Delta t$ and $(n-1)\Delta t$, respectively) are incorporated. We then end up at time $(n+3)\Delta t$ and $(n+1)\Delta t$ for Domain 1 and Domain 2, respectively. In order to step to $(n+1)\Delta t$ in Domain 2 field variables at intermediate times in Domain 1 are required. These additional values can be computed with time size $1\Delta t$ from time level $n\Delta t$ and $(n-1)\Delta t$ (open diamonds), and with time size $2\Delta t$ from time level $n\Delta t$ and $(n-2)\Delta t$ (open circles).
There is almost no extra numerical effort, since the most expensive term in the computations $-\mathbf{L^2}u^n$ is already available. Only extra memory locations need to be allocated

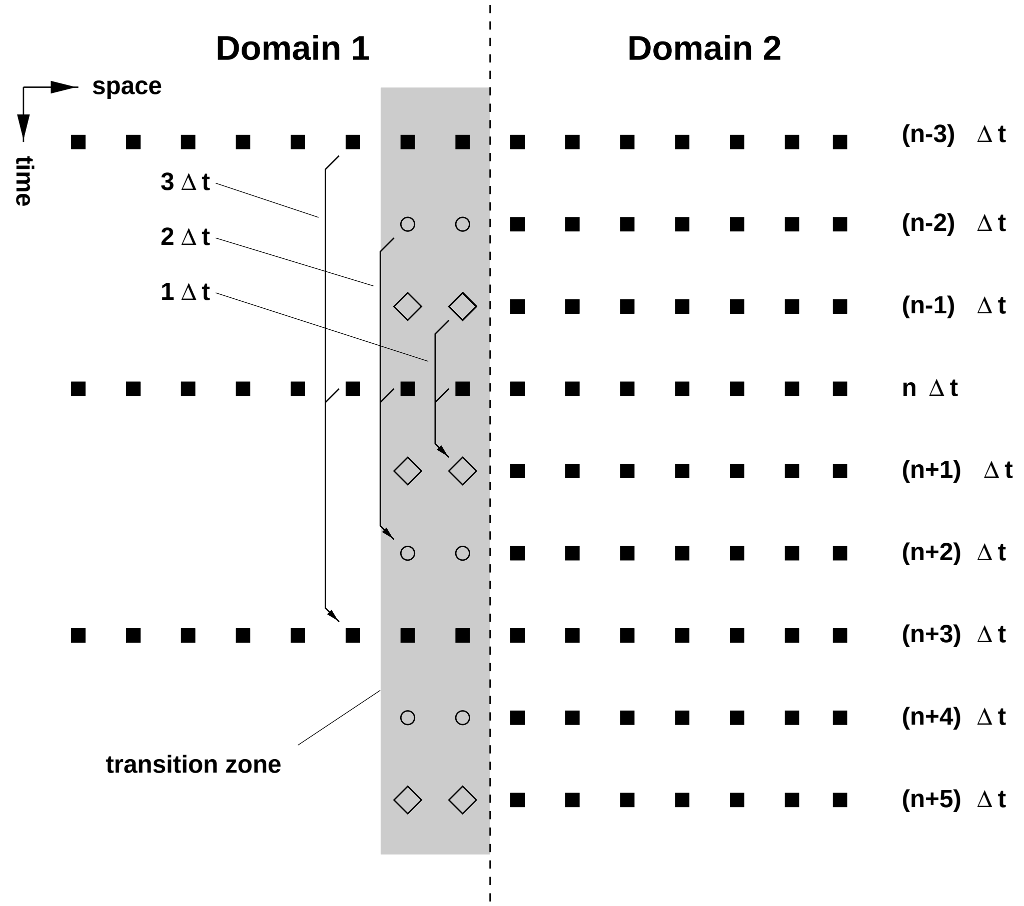

Figure 2: Time integration scheme, where time steps in Domain 2 (right) are three times smaller than in Domain 1 (left).

for keeping the field variables at intermediate time levels near the domain boundary (gray shaded area in Figure reftime).
If a staggered grid scheme is used the width of the transition zone needs to be do doubled, since two first order spatial derivatives are applied per time step.

EXAMPLES

Two examples demonstrate that the method with domain dependent time step size does not produce artifacts at the domain boundary and a comparison with an analytic solution shows its accuracy. The models are based on the 2-dimensional elastodynamic equations with 8th order staggered grid finite-difference derivative operators. The grid spacing, both in the horizontal and the vertical direction is 5 m. The source has a Ricker-like time history function with a cutoff frequency of 50 Hz.

Two halfspaces with large velocity contrast

The model is made up by two halfspaces (see Figure 3). The halfspaces are characterized by v_p=2000 m/s, v_s=1155 m/s and ρ=1000 kg/m^3 above the interface and by v_p=8000 m/s, v_s=4620 m/s and ρ=2000 kg/m^3 below the interface, respectively. This is a very strong contrast of the impedances. The explosive point source is positioned 100 m above the interface. The receivers R1, R2, and R3 are placed 220 m above the reflector. The horizontal distances against the source position are 300 m, 600 m, and 900 m. The time step size in the upper halfspace is 0.4 ms and in the lower halspace 0.1 ms, respectively. The ratio of the time step sizes reflect the contrast of the seismic velocities. The savings of computational costs due to different time steps were 37.5 percent, since the two domains were of same sizes. Figure 4 shows the comparison of the horizontal and vertical component of the displacements of the numerical and the analytical solutions, respectively. The analytical solutions was computed by a program of Berg et al. (1994). The comparison shows good agreement between the two solutions. However, the vertical component appears not as accurate as the horizontal component. Tests with a constant time step size of 0.25 ms throughout the entire model (not shown here) gave the same results. Therefore these small inaccuracies in the presence of large velocity contrasts must be attributed to staggered-grid FD methods in general.

Test for artificial reflections

In the second example the two halfspaces have exactly the same material parameters, so that there should be no physical reflection from the numerical interface. The parameters are v_p=2000 m/s, v_s=1155 m/s and ρ=1000 kg/m^3. The time steps in the

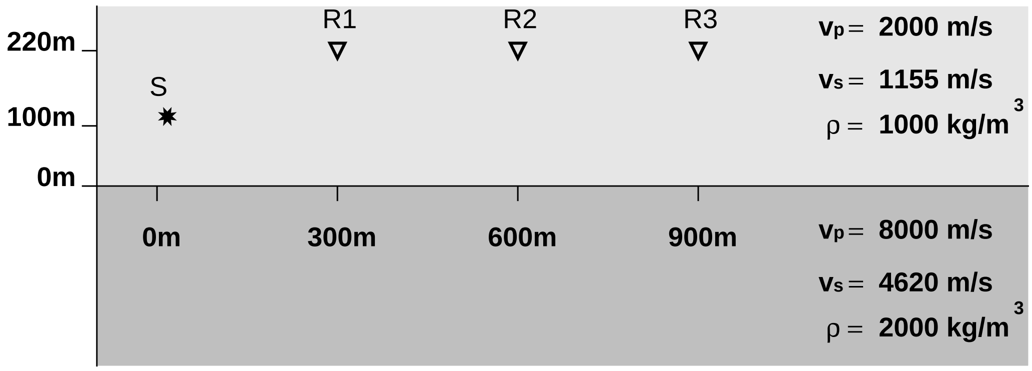

Figure 3: Source-receiver geometry and model parameters for the comparison with the analytic solution.

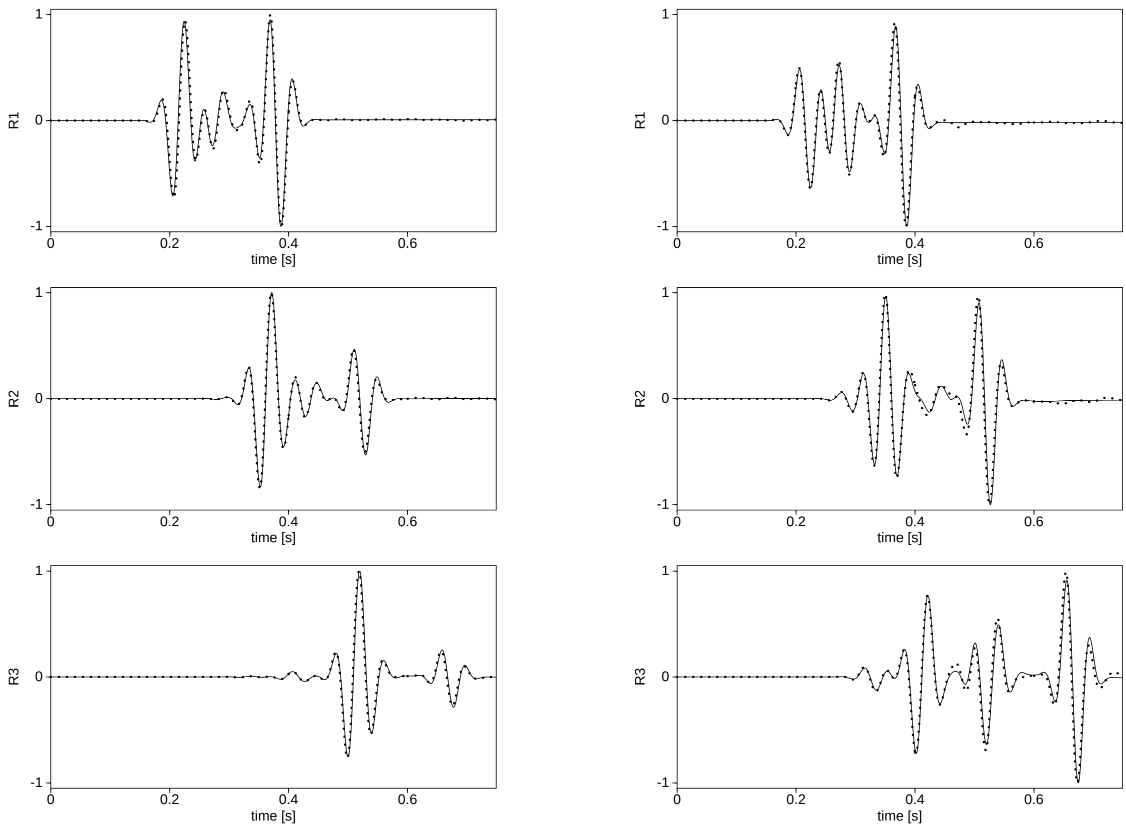

Figure 4: Comparisons of horizontal (left) and vertical (right) component of FD solution (dotted line) with analytic solutions (solid line) at receiver positions R1, R2, and R3 of Figure 3. All amplitudes are normalized.

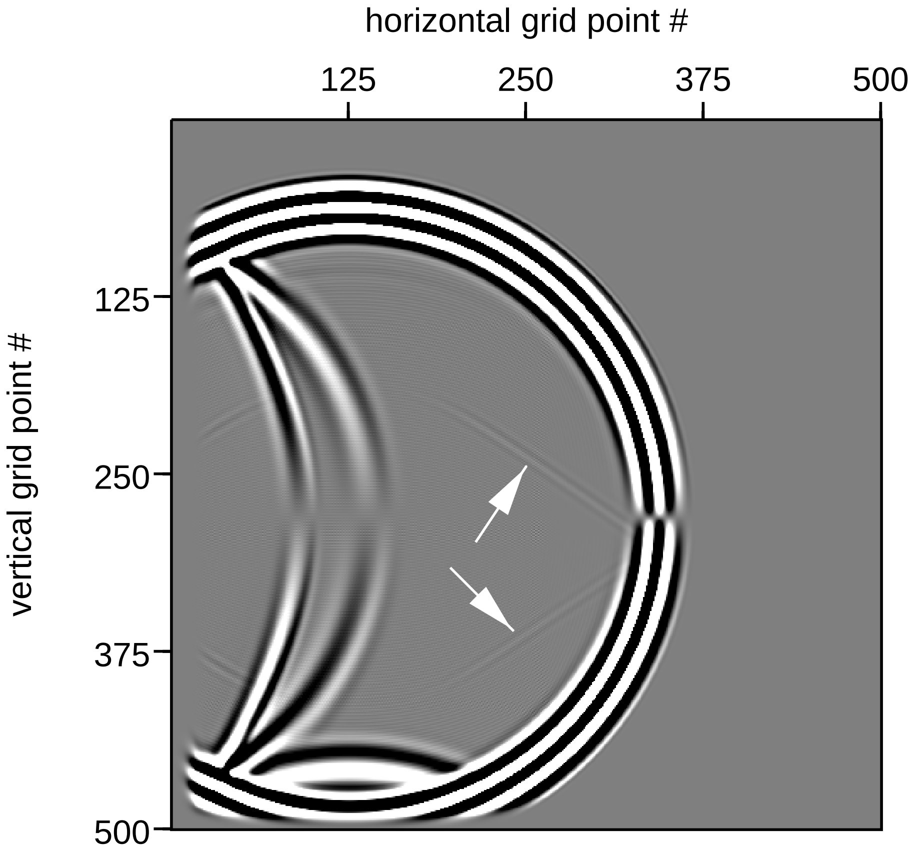

Figure 5: Vertical component snapshot of the displacement field in a homogeneous model at t=600 ms. The amplitudes are clipped at 1 percent of the maximum amplitude. Only a very small amount of artificial reflection/refraction from the numerical interface at vertical grid point # 300 can be observed.

lower halfspace are eight times smaller than in the upper halfspace. In order to make artifacts visible the gain was chosen such that amplitudes were clipped at 1 percent of the maximum value. The wavefield snapshot of the vertical displacement component shows that the wave front passes the numerical interface with only a negligible amount of reflected and refracted energy (see arrows in Figure 5). The amplitudes of the reflected/refracted arrival is much smaller than the arrival of the direct wave (see arrow in Figure 6). The reflection is caused by the different numerical behaviour of the two domains: due to the different time step sizes, the dispersion $\omega(k)$ in the two domains differ and hence the numerical velocities are slightly different. Reflections from the left side and the bottom are due to imperfectly absorbing boundary conditions.

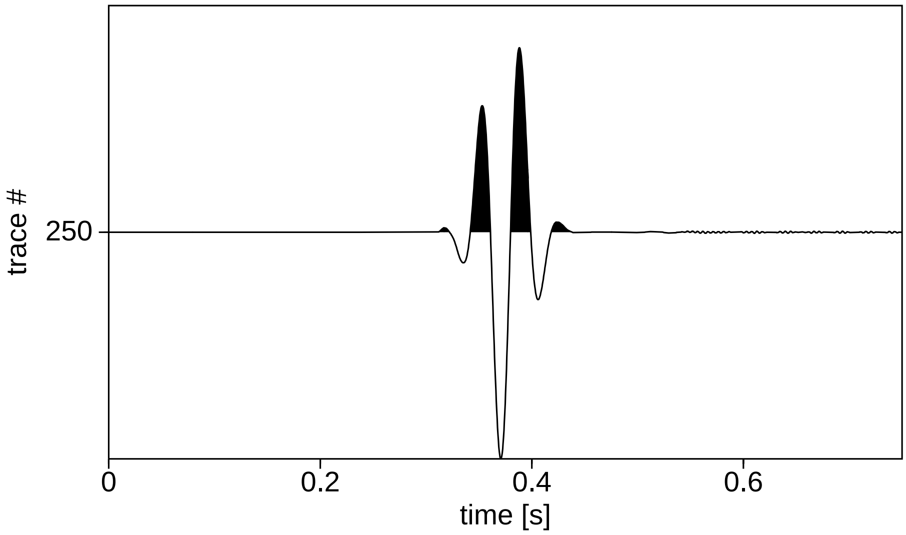

Figure 6: Vertical component of the displacement. The receiver position (compare Figure 5) is at grid point (250,250). Only the direct arrival can be seen. Artifacts are invisible.

CONCLUSIONS

Domain dependent time steps sizes in FD modeling can save computer time, if large velocity contrasts are present in the subsurface model. Also, if parts of the computational domain are represented on a very fine grid, this technique can save a considerable amount of computational work. The savings depend on the relative sizes of the computational domains and on the ratio of the respective time step sizes.
Comparisons with analytic solutions for two halfspaces have shown, that the technique works very accurate and that numerical artifacts are negligible. More than two domains with different time steps can be combined. The method can be extended straight forward to 3-dimensional modeling.

ACKNOWLEDGMENTS

The work was supported by Deutsche Forschungsgemeinschaft (German Science Foundation), Project Te-266/1-1.

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