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# Modeling of high contrasts in elastic media using a modified Finite Difference grid

*Erik H. Saenger, N.Gold and Sergei A. Shapiro*[1]

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## ABSTRACT

*The modeling of elastic waves with an explicit Finite Difference (FD) scheme on a staggered grid can cause instability problems when the medium possesses high contrast discontinuities. We have derived a new rotated staggered grid where all medium parameters are defined at appropriate positions within an elementary cell for the following operations. Using this grid it is possible to simulate the propagation of elastic waves in a medium containing cracks, pores or free surfaces without applying boundary conditions. In this report we show a synthetic example and compare the stability limit and the dispersion error for the new rotated staggered grid with the results of the standard staggered grid.*

## INTRODUCTION

An often used way of modeling the propagation of elastic waves are FD methods, discretizing the elastodynamic wave equation. They can be separated in time domain FD methods [Dablain (1986); Kneib and Kerner (1993)], using the 2nd order wave equation, and velocity stress methods, solving two coupled first order equations [Virieux (1986); Levander (1988)]. Using a staggered grid, where spatial derivatives are defined halfway between two grid points, both methods have the same locations of the modeling parameters within an elementary cell. A major disadvantage of a standard staggered grid is the fact that some modeling parameters are defined on inter grid locations, such that they have to be averaged or the grid values halfway have to be used. This leads to inaccurate results or instability problems when the propagation of waves in media with strong fluctuations of the elastic parameters (cracks, pores) is simulated.

[1] **email:** esaeng@gpirs5.physik.uni-karlsruhe.de

## DIFFERENT STAGGERED GRIDS

The FD modeling scheme bases on the discretization of the elastodynamic wave equation [Gubernatis et al. (1977)]:

$$\rho(\mathbf{r})\ddot{u}_i(\mathbf{r}) \quad = \quad (C_{ijkl}(\mathbf{r})u_{k,l}(\mathbf{r}))_{,j} + f(\mathbf{r})$$

Here, $u$ denotes the displacement, $C_{ijkl}$ denotes the stiffness tensor and $\rho$ is the density. In this equation we used standard tensor index convention. Indices that occur twice imply summation. The subscript $,j$ denotes differentiation with respect to the coordinate $j$.
Neglecting outer forces and boundary conditions, we have the following operations to calculate the acceleration $\ddot{u}$, starting from the wavefield :

- We calculate $\epsilon_{kl} = u_{k,l}$.
- We calculate the stress $\sigma_{ij} = C_{ijkl}u_{k,l}$.
- We calculate the divergence of the stress $\sigma_{ij,j}$.
- We calculate the acceleration $\ddot{u}_i = \frac{1}{\rho}\sigma_{ij,j}$.

The following step is the time update: calculating the wavefield at the next timestep from acceleration and the wavefield at the previous timesteps. This requires acceleration and the wavefield to be defined at the same position within the grid. On a staggered grid, spatial derivatives are located between the original points. This has the effect of much better results for numerical dispersion compared with centered grids, especially for the case of high-order spatial derivation operators. Considering the above operations, necessary for calculating the acceleration and taking into account the definition of a staggered grid, Virieux (1986) came up with the elementary cell for an isotropic elastic medium given in Figure 1 .

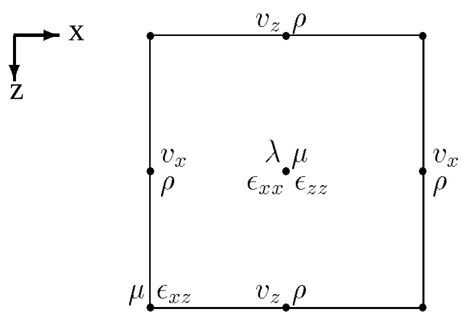



Figure 1: Locations, where strains, velocity (displacements), and elastic parameters are defined by Virieux (1986). Note, that the velocity components are not defined at the same location. $\lambda$ and $\mu$ denote the Lamé parameters.

We can see that within an elementary cell there are two density locations and two locations where the Lamé parameter $\mu$ is defined. Hence, we have to interpolate density and shear modulus. The density is defined at the corners but are needed halfway

each side of the cell. Therefore, they can be interpolated. The remaining problem is the accurate averaging of the shear modulus at the corners of the cell, and we have found no general solution of this problem in literature. An often used method [Kneib and Kerner (1993)] is to take the medium parameters of the cell for the side and corner locations, taking not into account that medium and wavefield parameters are multiplied which are not located at exactly the same position. This results directly in instability problems, if the medium contains very high contrasts.
The problems described above can be overcome if we find a grid, where all parameters are defined at positions so that no averaging of modeling parameters is necessary and no medium parameters from disadvantageous position have to be used. Starting with the assumption that all stiffness tensor elements are located at the same position, we arrive directly at the elementary cell of the new rotated staggered grid given in Figure 2 . In this grid, no averaging of elastic parameters is necessary. With this modified grid we can model a medium that contains very high contrasts of the elastic parameters and of the density.

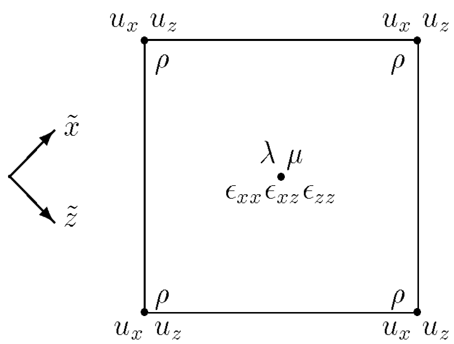



Figure 2: Locations where strains, displacements, density, and elastic parameters are defined on the new rotated staggered grid. Spatial derivations are performed along the $\tilde{x}$- and $\tilde{z}$-axes.

Considering the grid in Figure 2, the next step is to give up the limitation of isotropic media and of quadratic elementary cells. We find that all elements of the stiffness tensor are located at the position of $\lambda$ and $\mu$, therefore the grid can be applied to all kinds of anisotropy (up to triclinic).

## DISCRETIZATION OF DERIVATIVES

Since the wave equation contains derivatives in x- and z- direction and the modeling scheme contains derivatives along the $\tilde{x}$- and $\tilde{z}$-axes, we have to express derivatives along the old (x- and z-) axes in terms of derivatives along the new axes. For the new rotated staggered grid we find ($D_{\tilde{x}}$ and $D_{\tilde{z}}$ are the differentiation operators along the $\tilde{x}$- and $\tilde{z}$- directions):

$$\frac{\partial}{\partial z}u(z,x,t) \approx \frac{\sqrt{\Delta z^2+\Delta x^2}}{2\Delta z}(D_{\tilde{z}}u(z,x,t) - D_{\tilde{x}}u(z,x,t))$$
$$\frac{\partial}{\partial x}u(z,x,t) \approx \frac{\sqrt{\Delta z^2+\Delta x^2}}{2\Delta x}(D_{\tilde{z}}u(z,x,t) + D_{\tilde{x}}u(z,x,t))$$

## MODELING EXAMPLE

The modeling result given in Figure 3 we obtained using the new rotated staggered grid. Using a standard staggered grid [eg. Kneib and Kerner (1993)], this simulation could not be done because the modeling was unstable for high contrasts of the elastic parameters.

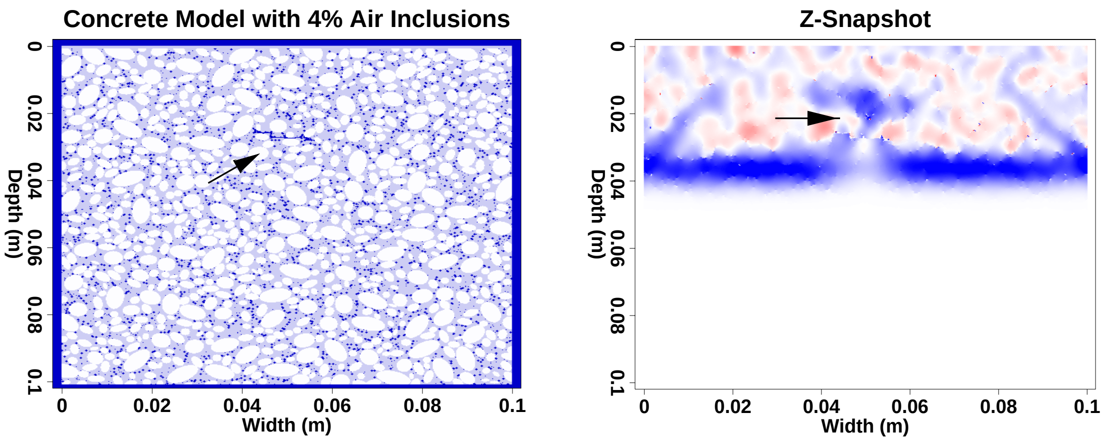



Figure 3: On the **left** side a heterogeneous concrete model with a defect, i.e. the crack, is shown. The ellipses represent gravel within cement. The concrete model is bounded by a thin layer of air. The **right** figure shows a Z-snapshot after a part of a plane wave has been reflected by the crack.

As medium we used a concrete model with a crack. Such a sample was used for non-destructive testing with ultrasound by other groups in our research project. At the interfaces of air-concrete we have very high contrasts of the elastic parameters and the density. The crack also consists of air. Modeling was done without applying any boundary conditions. We use a modified version of the Finite Difference program ULTIMOD [Karrenbach (1995)] to do the calculations with the new rotated staggered grid.

## STABILITY AND DISPERSION

Since FD modeling approximates derivatives by numerical operators and uses Taylor polynomials to perform the time update, inaccuracies occur, especially for coarse grids. One can separate these numerical errors into amplitude and phase errors. In principle, for a plane wave propagating through a homogeneous medium, the amplitude must be conserved, and the velocity of propagation should not be frequency-dependent. In FD modeling, it is possible that the amplitude increases exponentially with every timestep. In this case, the modeling scheme is said to be unstable. Velocity errors, also called numerical dispersion, can not be excluded completely but can be estimated and, therefore, reduced to a known and acceptable degree. The goal of this section is to give the stability limit (von Neumann, for more details see eg. Crase (1990)), and to show the dispersion error in the case of $8^{th}$ order spatial derivatives and a $2^{nd}$ order time update. The results for the standard staggered grid [eg. Kneib and Kerner (1993)] and for the new rotated staggered grid are given in order to compare in section  and in section .

### Results for the standard staggered grid

The stability limit for the **standard** staggered grid [eg. Kneib and Kerner (1993)] is (2D, 2nd order in time, h := grid spacing, $a_n$ := coefficients of the space operator):

$$v_{max}\Delta t/h \leq 1/(\sqrt{2}\sum_n |a_n|) \qquad (\sigma = 100\%)$$

Figure 4: Relative accuracy of phase velocity for the **standard** staggered grid (2D). The propagation angle relative to the grid is **0°**. Curves are computed with second order in time and 8th order in space. The value of $\sigma$ denotes the percentage of the respective stability limit. The vertical dashed line shows the limit of the dispersion criterion.

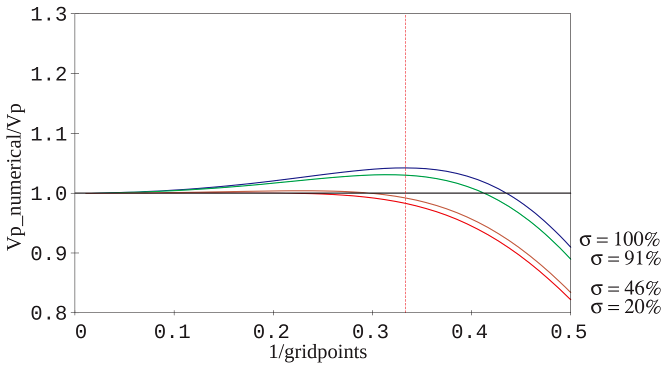
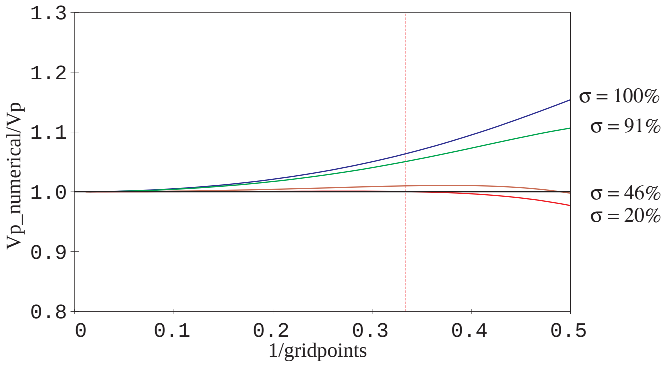



Figure 5: Relative accuracy of phase velocity for the **standard** staggered grid. The propagation angle relative to the grid is **45°**. The value of $\sigma$ varies from 100% (top) over 91% and 46% to 20% (bottom).




To avoid a relative error in phase velocity of more than 1%, when using the **standard** staggered grid, one has to satisfy the following two criteria (2D, 2nd order in time, 8th order in space, Central Limit coefficients [Karrenbach (1995); Kindelan et al. (1990)]):

1. Stability criterion (**standard** staggered grid):

$$v_{max}\Delta t/h < 0.25 \approx 46\% \times 1/(\sqrt{2}\sum_n |a_n|) \quad (\Rightarrow\ \sigma = 46\%)$$

2. Dispersion criterion (**standard** staggered grid):

$$\frac{v_{min}}{f_{max}}/h > 3 \quad (see\ the\ dashed\ line\ in\ Fig.4\ and\ in\ Fig.5)$$

## Results for the new rotated staggered grid

The stability limit for the new **rotated** staggered grid is (2D, 2nd order in time, h := grid spacing, $a_n$ := coefficients of the space operator):

$$v_{max}\Delta t/h \leq 1/(\sum_n |a_n|) \qquad (\sigma = 100\%)$$

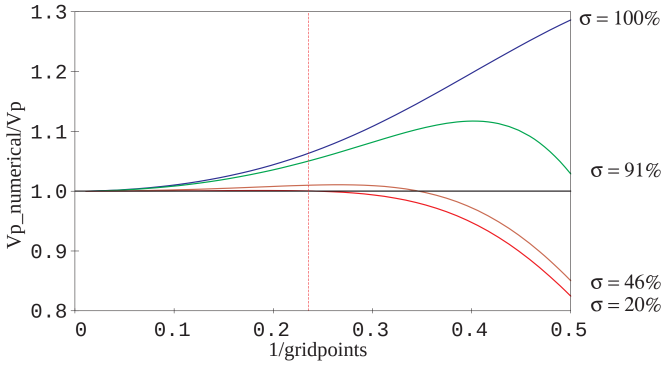



Figure 6: Relative accuracy of phase velocity for the new **rotated** staggered grid (2D). The propagation angle relative to the grid is **0°**. Curves are computed with second order in time and 8th order in space. The value of $\sigma$ denotes the percentage of the respective stability limit. The vertical dashed line shows the limit of the dispersion criterion.

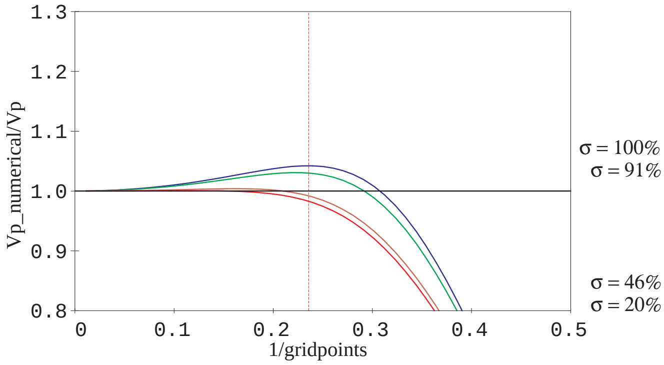



Figure 7: Relative accuracy of phase velocity for the new **rotated** staggered grid. The propagation angle relative to the grid is **45°**. The value of $\sigma$ varies from 100% (top) over 91% and 46% to 20% (bottom).

To avoid a relative error in phase velocity of more than 1%, when using the new **rotated** staggered grid, one has to satisfy the following two criteria (2D, 2nd order in time, 8th order in space, Central Limit coefficients [Karrenbach (1995); Kindelan et al. (1990)]):

1. Stability criterion (new **rotated** staggered grid):

$$v_{max}\Delta t/h < 0.25\sqrt{\mathbf{2}} \approx 46\% \times 1/(\sum_n |a_n|) \quad (\Rightarrow\ \sigma = 46\%)$$

2. Dispersion criterion (new **rotated** staggered grid):

$$\frac{v_{min}}{f_{max}}/h > 3\sqrt{\mathbf{2}} \quad (see\ the\ dashed\ line\ in\ Fig.6\ and\ in\ Fig.7)$$

## CONCLUSION

The grid modifications shown in this report enable modeling of arbitrary inhomogeneities without boundary conditions in two and three dimensions, as well as in all kinds of anisotropic media. The dispersion analysis shows that there is no essential difference in phase velocity error between the standard staggered grid and the new rotated grid. Until now, no limitations concerning applications of the modified grid have been found by the authors. Thus, a major improvement of FD modeling has been achieved.

## REFERENCES


Crase, E., 1990, High-order (space and time) finite-difference modelling of the elastic wave equation: 60th Ann. Internat. Mtg.,Soc. Expl. Geophys.,Expanded Abstracts, pages 987–991.

Dablain, M. A., 1986, The application of high-order differencing to the scalar wave equation: Geophysics, **51**, 54–66.

Gubernatis, J. E., Domany, E., Krumhansl, J. H., and Huberman, M., 1977, The born approximation in the scattering of elastic waves by flaws: J. appl. Phys., **48**, 2812–2819.

Karrenbach, M., 1995, Elastic tensor wavefields: Ph.D. thesis, Stanford University.

Kindelan, M., Kamel, A., and Sguazzero, P., 1990, On the construction and efficiency of staggered numerical differentiators for the wave equation: Geophysics, **55**, 107–110.

Kneib, G., and Kerner, C., 1993, Accurate and efficient seismic modelling in random media: Geophysics, **58**, 576–588.

Levander, A. R., 1988, Fourth-order finite-difference p-sv seismogramms: Geophysics, **53**, 1425–1436.

Virieux, J., 1986, Velocity-stress finite-difference method: Geophysics, **51**, 889–901.