Wave Inversion Technology, Report No. 2, pages 171-179

Traveltime computation for 3D anisotropic media

N. Ettrich[1]

ABSTRACT

We present a method for computing first-arrival traveltimes in three dimensional (3-D) anisotropic media. The basic routine is a finite-difference (FD) eikonal solver for elliptically anisotropic media. It allows for arbitrary orientation of the tensor ellipsoid and for strong anisotropy. To achieve stability a wavefront expansion scheme is applied. Second-order approximations of the eikonal equation make the FD eikonal solver highly accurate with relative errors of the order of a few permille for homogeneous models. The method is stable also in models with strong velocity gradients. A perturbation scheme for a first-order correction of traveltime is included. It allows to consider media with general anisotropy.

INTRODUCTION

Since anisotropy has been recognized as an important feature of seismic wave propagation there is an interest of extending methods of exploration seismology to anisotropic media. Prestack Kirchhoff-type migration and traveltime tomography require the fast computation of traveltimes which is still a challenge in 3-D even for isotropic media. For computational efficiency we develop a FD eikonal solver for which several strategies were derived in the past (for 3-D isotropic media, e.g., Vidale (1988)). For 2-D anisotropic media methods by, e.g., Lecomte (1993) and Dellinger and Symes (1997) (with extension to 3-D) exist. Anisotropic wave propagation is a 3-D process. The eikonal equation generally does not factorize into lower order eikonal equations for different types of waves. In the framework of FD methods 6th-order polynomials for components of slowness vectors have to be solved numerically which is slow and may lead to instabilities. In contrast, in elliptically anisotropic media the eikonal equation for the considered wave type is only slightly more complex compared to the isotropic case, and all required relations are available in closed form. Elliptical anisotropy is of limited physical significance. The FD code for elliptically anisotropic media can, therefore, be understood as a basic routine for computation of traveltimes in arbitrarily anisotropic media. If the deviation between an elliptically anisotropic model and the model with given anisotropy allows for a linearization of traveltimes, a computation

[1] **email:** ettrich@dkrz.de

by a first-order perturbation method embedded into the FD scheme is possible Ettrich and Gajewski (1998).

BASIC FORMULAS

An elliptically anisotropic medium is defined by phase velocities $\tilde{v}_h$ and $\tilde{v}_v$ in horizontal and vertical direction and two angles γ_1 and γ_2 describing the orientation of the crystal coordinate system with respect to the global Cartesian coordinate system. Here, directions and all quantities marked with the $\sim$-sign refer to the crystal coordinate system. The slowness surface (eikonal equation) for the slowness vector $(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z)$ reads

$$\tilde{v}_h^2 \tilde{p}_x^2 + \tilde{v}_h^2 \tilde{p}_y^2 + \tilde{v}_v^2 \tilde{p}_z^2 = 1 \tag{1}$$

and allows for the computation of one slowness vector component if the others are known and for the computation of phase velocity for a given phase direction because the absolute value of the slowness vector equals inverse phase velocity. In the crystal coordinate system no azimuthal dependence upon rotation around the $\tilde{z}$-axis exists. Therefore, the angle of inclination $\tilde{\alpha}_{\text{ph}} = \text{atan}(\sqrt{\tilde{p}_x^2 + \tilde{p}_y^2}/\tilde{p}_z)$ is sufficient to describe the position at the slowness surface. The corresponding direction $\tilde{\alpha}_{\text{gr}}$ and absolute value $\tilde{v}_{\text{gr}}$ of the group velocity vector are simply:

$$\tan\tilde{\alpha}_{\text{gr}} = \frac{\tilde{v}_h^2}{\tilde{v}_v^2}\tan\tilde{\alpha}_{\text{ph}}\ ,\ \tilde{v}_{\text{gr}} = \frac{\tilde{v}_{\text{ph}}}{\cos\left(\tilde{\alpha}_{\text{gr}} - \tilde{\alpha}_{\text{ph}}\right)}. \tag{2}$$

However, some stencils of the program work on the eikonal equation in the global Cartesian coordinates where eq. (1) is transformed to:

$$Ap_x^2 + Bp_y^2 + Cp_z^2 + 2Dp_xp_y + 2Ep_xp_z + 2Fp_yp_z = 1. \tag{3}$$

Coefficients A to F are functions of γ_1, γ_2, $\tilde{v}_h$, and $\tilde{v}_v$.

FD APPROXIMATION OF THE EIKONAL EQUATION

In 3-D preferably a coarse grid should be used. Therefore, we are looking for a second-order approximation of the eikonal equation. While in 2-D a second-order formula can be derived by determining the first-order expressions for the slowness vector components and inserting them into the 2-D eikonal equation (Vidale (1988)) we follow here a different strategy. Similar to Vidale (1990) (his eq. 2) we make an ansatz for the searched traveltime t_9 at the corner point of a cubical cell (see Fig. 1, traveltimes known at points 3–8) where the squared difference $(t_9 - t_0)^2$ is given by (Leidenfrost and Gajewski (1998))

$$\begin{aligned}(t_9 - t_0)^2 = \quad & w_0 + w_1[(t_5 - t_3)^2 + (t_7 - t_8)^2]+ \\ & w_2[(t_5 - t_4)^2 + (t_6 - t_8)^2]+ \\ & w_3[(t_6 - t_3)^2 + (t_7 - t_4)^2]+ \\ & w_4[(t_3 - t_4)^2 + (t_6 - t_7)^2]+ \\ & w_5[(t_6 - t_5)^2 + (t_8 - t_4)^2]+ \\ & w_6[(t_7 - t_5)^2 + (t_8 - t_3)^2] \ .\end{aligned} \tag{4}$$

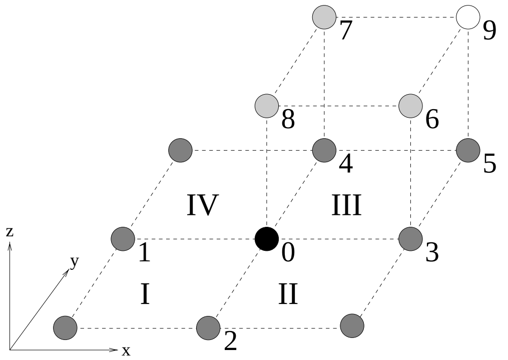

Figure 1: Computation starting at point 0.

To determine the weights w_i, $i = 1, \cdots, 6$, eq. (4) must be satisfied for plane waves with normal vectors in x-, y-, and z-direction (directions 1,2,3) and in direction of the diagonals x-y, x-z, and y-z (directions 4, 5, 6). We, therefore, calculate traveltimes t_k at points (x_k, y_k, z_k), $k = 3, \cdots, 9$, using the plane wave formula

$$n_x^{(j)}(x_k - x_0) + n_y^{(j)}(y_k - y_0) + n_z^{(j)}(z_k - z_0) = v_j t_k \tag{5}$$

for plane waves with normal vectors $(n_x^{(j)}, n_y^{(j)}, n_z^{(j)})$ in each of these six directions j, $j = 1, \cdots, 6$. The phase velocity v_j for direction j is obtained by transforming $(n_x^{(j)}, n_y^{(j)}, n_z^{(j)})$ into the crystal coordinate system and solving eikonal equation (1). Since eq. (4) is an approximation for the eikonal equation all quantities used here refer to the phase velocity, not the group velocity vector. Inserting the plane wave traveltimes for each of the six directions into ansatz (4) and solving the system of linear equations for the six unknowns w_i we find:

$$\begin{aligned} w_1 &= 1.5 - \tfrac{w_0}{4h^2}(2v_4^2 + 2v_6^2 - v_1^2 - v_3^2) \\ w_2 &= 1.5 - \tfrac{w_0}{4h^2}(2v_4^2 + 2v_5^2 - v_2^2 - v_3^2) \\ w_3 &= 1.5 - \tfrac{w_0}{4h^2}(2v_5^2 + 2v_6^2 - v_1^2 - v_2^2) \\ w_4 &= \tfrac{w_0}{4h^2}(2v_4^2 - v_1^2 - v_2^2) - 0.5 \\ w_5 &= \tfrac{w_0}{4h^2}(2v_6^2 - v_2^2 - v_3^2) - 0.5 \\ w_6 &= \tfrac{w_0}{4h^2}(2v_5^2 - v_1^2 - v_3^2) - 0.5, \end{aligned} \tag{6}$$

where h denote grid spacing and $w_0 = 3h^2/v_7^2$. Velocity v_7 is the phase velocity in direction of the big diagonal from point 0 to point 9.

Formula (4) with coefficients (6), called stencil 1, is applied to the majority of grid points. It is applicable if seven grid points of the actual grid cell are already timed.

SCHEME OF EXPANSION

To retain causality and to guarantee stability we expand wavefronts (Qin et al. (1992)) rather than cubes (Vidale (1990)). This is achieved by ordering the outer points of the irregular volume of timed points with respect to traveltime from minimum to maximum. In each step, the first point of this ordered traveltime array is picked to compute traveltimes to adjacent points. Actually timed points are inserted into the traveltime array due to their traveltime, and in the next step this procedure is repeated. If point 0 (see Fig. 1) is such a point of minimum traveltime and if points 1, 2, 3, and 4 are also already timed the traveltime to point 8 can be computed and consecutively to point 6. Formulas of Vidale (1990) (his formulas 3 and 4) are modified for this purpose to solve the eikonal equation (3). To compute traveltime t_8 slowness vector components p_x and p_y are approximated by $p_x = (t_3 - t_1)/(2h)$ and $p_y = (t_4 - t_2)/(2h)$, and the eikonal equation (3) is solved for the remaining component p_z. Then, t_8 is $t_0 + hp_z$, with grid spacing h. This is the formula with lowest accuracy (stencil 3). Traveltime t_6 is obtained by approximating $p_y = (t_4 - t_2)/(2h)$, $p_x = (t_3 - t_0 + t_6 - t_8)/(2h)$ and $p_z = (t_8 - t_0 + t_6 - t_3)/(2h)$. With theses expressions for the slowness vector components the eikonal equation (3) is solved for the searched traveltime t_6 (stencil 2).

ACCURACY AND STABILITY OF THE METHOD

The FD approximation (4) together with (6) is based on a plane wave concept inside the cubical cells of the grid. Using a homogeneous model we check accuracy for curved wavefronts. Horizontal velocity is 2 km/s, vertical velocity is 2.4 km/s. The tensor is rotated by 30° in azimuth and 30° in inclination. Grid spacing is 20 m. A cubical region of five grid points in each direction around the source is initialized using formulas (1), and (2). Fig. 2 (left-hand side) displays numerically computed wavefronts. Accuracy is quantified in its right-hand side where the maximum relative error does not exceed 0.2%. While the error is largest close to vertical and horizontal direction of propagation it considerably decreases towards the diagonals and big diagonals.

Further, a horizontally layered model is considered. Parameters of the upper layer equal parameters of the homogeneous model considered above (anisotropy amounts to 20%). In a depth of 0.66 km, both, horizontal and vertical velocity (with respect to the crystal coordinate system) increase by 1.2 km/s, and in a depth of 1.0 km they increase by 1.0 km/s, and 0.8 km/s, respectively. Moreover, the tensor ellipsoid is rotated by additional 25° in azimuth here. Wavefronts in Fig. 3 demonstrate the stability of the method for such models with strong velocity contrasts. At both interfaces head waves are generated. Weakly smoothing the model with a central operator of the width of the grid spacing eliminates the small oscillations of the wavefronts.

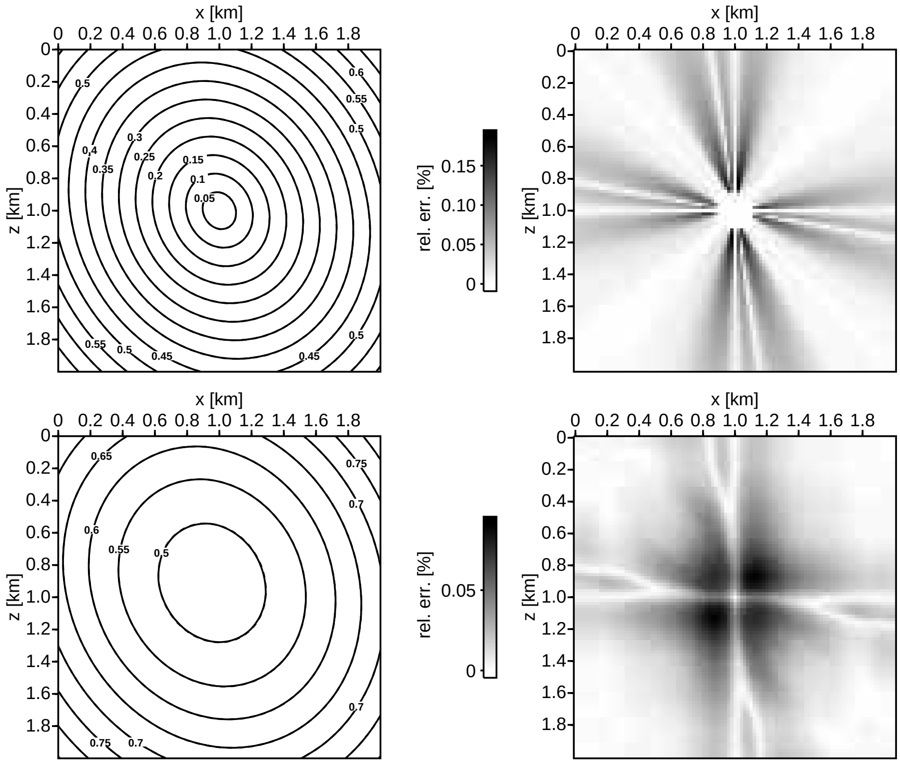

Figure 2: Left: Wavefronts in a homogeneous elliptically anisotropic model; x-z-slices with offset y=0 km (upper), and 1 km (lower) from source. Right: Corresponding relative errors.

TRAVELTIME PERTURBATION

To compute traveltimes in arbitrarily anisotropic media a perturbation scheme is introduced into the FD eikonal solver. We consider a model with arbitrary anisotropy as a perturbed model with respect to an elliptically anisotropic reference model. To first-order the difference of traveltime between two fixed points S and P between both models is given by (Cerveny and Jech (1982)):

$$\Delta t(P,S) = -\frac{1}{2}\int_{t(S)}^{t(P)} \Delta a_{ijkl} p_i p_l g_j g_k dt. \tag{7}$$

All quantities, i.e., slowness vector p_i, polarization vector g_i and differences of elastic parameters Δa_{ijkl} between both models are taken along the ray between S and P in the elliptically anisotropic reference model where polarization vectors are available in closed form. Since FD reference traveltimes are directly computed at grid points rays must be approximated to carry out the integration. Following the local plane wavefront assumption the associated ray segments inside the cells are straight. We, therefore, simply multiply the integrand of eq. (7), which is constant for each cell, with the traveltime difference between reference traveltime t_{ref} at point 7 and t_{ref} at point Q (see Fig. 4) and add this contribution of the cell to Δt at Q. Point Q is the intersection of the straight ray segment with the boundary of the cubical cell. The direction of the ray segment is given by eq. (2) with the angle of phase velocity obtained by finite differencing reference traveltimes at points $0-7$. Δt at Q is linearly interpolated from Δt at points $0-3$, and, finally, traveltime t_{pert} at point 7 in the anisotropic model is simply $t_{\text{pert}} = t_{\text{ref}} + \Delta t$.

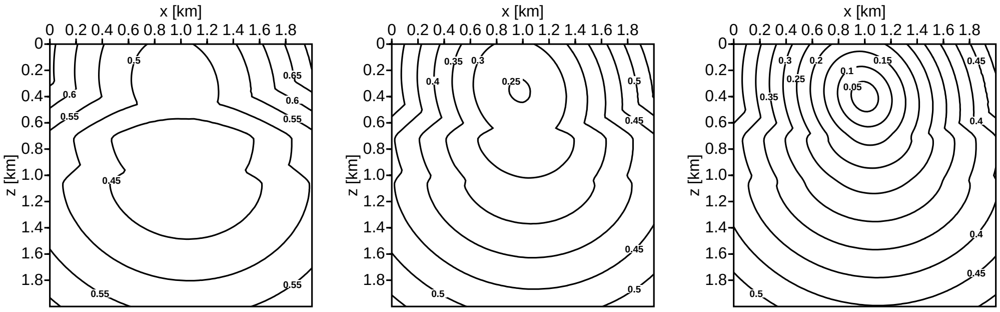

Figure 3: Wavefronts in a 3-D elliptically anisotropic inhomogeneous model; x-z-slices with offset y=0 km (left), 0.5 km (middle) and 1 km (right) from source.

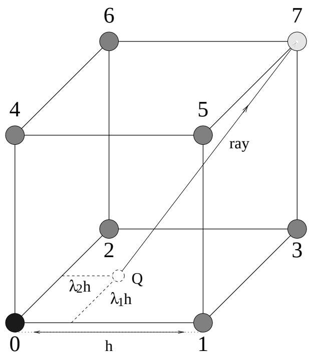

Figure 4: Cubical grid cell. Local straight ray segment reaching point 7 intersects boundary of cube in point Q.

Accuracy of the 3-D FD perturbation method is demonstrated in Fig. 5. The model consists of two orthorhombic layers with coefficients A_{11}=4.0, A_{12}=1.5, A_{13}=2.3, A_{22}=4.5, A_{23}=1.9, A_{33}=6.0, A_{44}=1.0, A_{55}=1.1, A_{66}=1.4 (km^2/s^2) for the first layer and A_{11}=6.0, A_{12}=1.3, A_{13}=2.6, A_{22}=6.5, A_{23}=2.9, A_{33}=8.0, A_{44}=2.0, A_{55}=2.2, A_{66}=2.5

(km^2/s^2) for the second layer. All other coefficients equal zero. Traveltime curves for a surface source at x=y=0.5 km at three vertical receiver lines at 1. x=0.6 km, y=0.5 km, 2. x=1.0 km, y=0.5 km, and 3. x=1.0 km, y=1.0 km are displayed, reduced with a velocity of 3 km/s. The relative error for traveltimes computed with the FD perturbation method (short dashes, compare to traveltimes obtained with a ray tracer for anisotropic media, Gajewski and Psencik (1990), solid lines) is everywhere less than 1%.

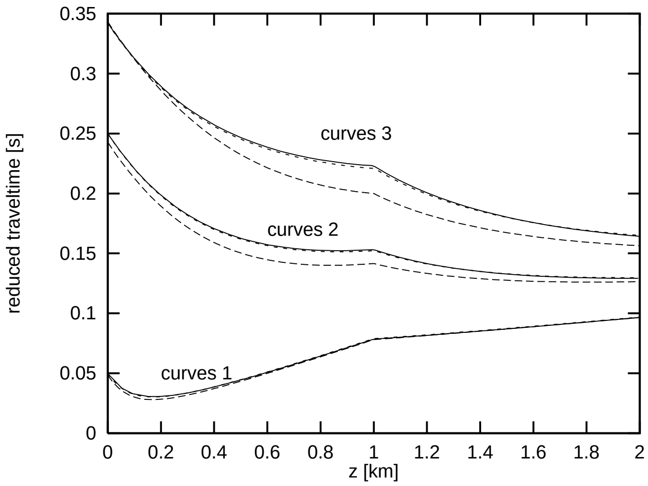

Figure 5: Curves of traveltime in an orthorhombic medium, computed with a ray tracer for anisotropic media (solid lines), and computed with the FD perturbation method (short dashes); reference traveltimes for the elliptically anisotropic reference medium (long dashes).

CONCLUSIONS

The presented algorithm provides a method for the efficient computation of first-arrival traveltimes in 3-D elliptically anisotropic media. A first-order perturbation scheme is included to consider arbitrarily anisotropic media. Future work should be devoted to the determination of best-fitting elliptically anisotropic reference media.

REFERENCES

Cerveny, V., and Jech, J., 1982, Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media: J. Geophys., **51**, 96–104.

Dellinger, J., and Symes, W., 1997, Anisotropic finite-difference traveltimes using a hamilton-jacobi solver: Presented at the 67th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts.

Ettrich, N., and Gajewski, D., 1998, Traveltime computation by perturbation with fd-eikonal solvers in isotropic and weakly anisotropic media: Geophysics, **63**, 1066–1078.

Gajewski, D., and Psencik, I., 1990, Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures: J. Geophys. Res., **95**, 11301–11315.

Lecomte, I., 1993, Finite difference calculation of first traveltimes in anisotropic media: Geophys. J. Int., **113**, 113–342.

Leidenfrost, A., and Gajewski, D., 1998, A 3-d fd eikonal solver for non-cubical grids: Presented at the 68th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts.

Qin, F., Luo, Y., Olsen, K. B., Cai, W., and Schuster, T. G., 1992, Finite-difference solution of the eikonal equation along expanding wavefronts: Geophysics, **57**, 478–487.

Vidale, J. E., 1988, Finite-difference calculation of traveltimes: Bull. Seis. Soc. Am., **78**, 2062–2076.

Vidale, J. E., 1990, Finite-difference calculation of traveltimes in three dimensions: Geophysics, **55**, 521–526.