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Reciprocal volume and surface scattering integrals for anisotropic elastic media

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ABSTRACT

Linear scattering theory in anisotropic media is useful for describing modeling, inversion and migration algorithms. The single-scattering or Born approximation leads to a volume scattering integral which is further simplified by using the geometrical ray approximation (GRA) for the Green's functions from the source and receiver to the scattering point.

Discontinuities of the medium parameters which are confined to smooth surfaces will reflect and refract the propagating waves. This is often described by the Kirchhoff-Helmholtz integral, which uses the Green's representation of the reflected field and the Kirchhoff approximation for the field and its normal derivative at the surface. The reflected field and its derivative are often approximated by multiplying the corresponding parts of the incoming field with the plane-wave reflection coefficient computed for the angle between the incoming ray and the surface normal (Kirchhoff approximation). Besides the inconsistency of imposing both the field and its normal derivative on the surface to represent the field away from it, the Kirchhoff-Helmholtz integral gives rise to a reflected response which is non-reciprocal.

The Born and Kirchhoff-Helmholtz integrals have traditionally been treated as completely separate formulations in the studies of reflection and transmission of waves due to smooth interfaces. A simple use of the divergence theorem applied to the Born volume integral gives a reciprocal surface scattering integral, which can be seen as a natural link between the two formulations. This unifying integral has been recently derived in the context of inversion. We call it the Born-Kirchhoff-Helmholtz (BKH) integral.

The properties of the BKH integral are investigated by a stationary-phase evaluation, and the result is interpreted in ray theoretical terms. For isotropic media, explicit expressions are given.

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INTRODUCTION

Linear scattering theory provides a suitable framework for describing modeling, inversion and migration algorithms (Bleistein, 1987; Beylkin and Burridge, 1990; Schleicher et al., 1993; Chapman and Coates, 1994; Spencer et al., 1995; Burridge et al., 1995; de Hoop and Bleistein, 1996). The parameters of the medium are split into two parts corresponding to a background or reference medium and a perturbed medium. The parameters of the reference medium are usually smoothly varying functions, while the perturbations are considered to act as point scatterers or reflecting or refracting interfaces. Assuming single scattering, the Born approximation is readily derived.

By using the geometric ray approximation (GRA) of the Green's functions in the reference medium, the Born approximation provides a standard volume integral (Wu, 1989; Beylkin and Burridge, 1990). Asymptotic ray theory for anisotropic media is described in detail in Cervený (1985) or, in more compact form in Chapman and Coates (1994).

The discontinuities of the medium parameters are often located on smooth surfaces which reflect and refract the propagating waves. The response from such an interface may be formulated as a surface scattering integral. The classical approach is the use of the Kirchhoff-Helmholtz integral (Pao and Varatharajuli, 1976; Frazer and Sen, 1985; Tygel et al., 1994; Goldin, 1991). This is the result of the application of Green's representation theorem to the reflected field as a surface integral involving the Green's function of the reflected field and its normal derivative. The reflected field and its derivative are approximated by the corresponding parts of the incoming field, multiplied by the plane-wave reflection coefficient that pertains to the angle between the incoming ray and the surface normal. This is the so-called Kirchhoff approximation (see, e.g., Bleistein, 1984).

Although very widely used in practice for the construction of synthetic seismograms, as well as for inversion schemes to localize the reflector and to determine other attributes of the reflector, the Kirchhoff-Helmholtz integral presents several inconsistencies. First, is not correct mathematically as the representation theorem from which it is based does not permit to assign arbitrary both the field and its normal derivative independently. Second, the obtained expression is non-reciprocal, which violates a fundamental physical property of the reflected field.

In the same way as de Hoop and Bleistein (1996), we use a more general version of the divergence theorem (see Bleistein, 1984, equation (2.82)) to transform the Born volume integral into a surface scattering integral. We find it instructive to prove this important relationship in two simple ways. The first method is to approximate the divergence volume scattering integral along surfaces that are normal to the gradient of the steeply varying parameters of the medium. This is similar to a technique which was used by Ursin and Haugen (1996) to derive the plane-wave scattering matrix in anisotropic media. Secondly, the volume scattering integral is divided into two parts, where the two new volumes are separated by the reflecting interface. The parameters in these two parts are represented by a smoothly varying reference medium and two perturbations, each of which is also a smooth function. The new integral now involves the jumps in the parameters and geometrical factors at the interface. This parameterization is similar to the one used by Banik (1987) and Haugen and Ursin (1996) for approximating the plane-wave scattering matrix. One may also use the average parameters of the two media in contact, to define the background medium (Aki and Richards, 1980; Ursin and Haugen, 1996).

Properties of the new integral surface scattering can be derived by evaluating the integral by the method of stationary phase and comparing the result with the GRA for the reflected wave. In particular, this provides a relationship between the point-scattering coefficients and the linearized plane-wave reflection coefficients, as given by Beylkin and Burridge (1990) for a PP-reflection in isotropic media. The complete results for isotropic media have also been summarized. Since the scattering coefficients are reciprocal, they must be related to the plane-wave reflection coefficients, which have been normalized with respect to the normal energy flux (Chapman, 1994). Reflection coefficients normalized with respect to amplitude do not satisfy reciprocity.

Furthermore, comparing the stationary-phase approximation and the GRA solution, gives a relation between the total geometrical spreading from the source to the reflection point and to the receiver, and the geometrical spreading factors from the source and receiver to the reflection point. A relation is also obtained for the phase shifts due to caustics along the reflection ray. These relations have been obtained by Goldin (Goldin, 1991) (1991) and by Schleicher et al. (Schleicher et al., 1993) (1993) for isotropic media.

The scattering integrals are all derived assuming single scattering which is valid only for small perturbations of the parameters of the medium. Since the expressions are linear functions of the perturbations, they produce independent contributions to the scattered field. In a complicated medium, there may then be contribution both from point scatterers and reflecting/refracting interfaces. We also consider the GRA Green's function for a single wave type, but with possible mode conversion from the source to the scattering point to the receiver. The total scattered field is, of course, a sum of such waves (Chapman and Coates, 1994; Cervený, 1985).

CONCLUSION

We have reviewed the Born scattering theory in anisotropic media and expressed the scattered field as a volume integral using the GRA Green's functions. By applying the divergence theorem, this was changed into a volume scattering integral involving the gradient of the elastic parameters.

For media with large gradients or discontinuities in the parameters, we derived reciprocal surface scattering integrals. The stationary-phase approximation of these integrals gave a relationship between the scattering coefficients and the linearized, weak-contrast plane-wave reflection coefficients, which have been normalized with respect to the normal energy flux. For isotropic media, explicit expressions can be found in the Appendix.

Comparison of the GRA Green's function for a reflected wave and the asymptotic value of the new surface integral resulted in a relationship between the complex amplitude function of the reflected wave and the amplitude functions of the waves from the source to the reflection point and the receiver to the same point. This involved the Fresnel matrix, which is the second-derivative matrix of the total traveltime with respect to the surface coordinates.

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PUBLICATIONS

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